Absolute Root Separation

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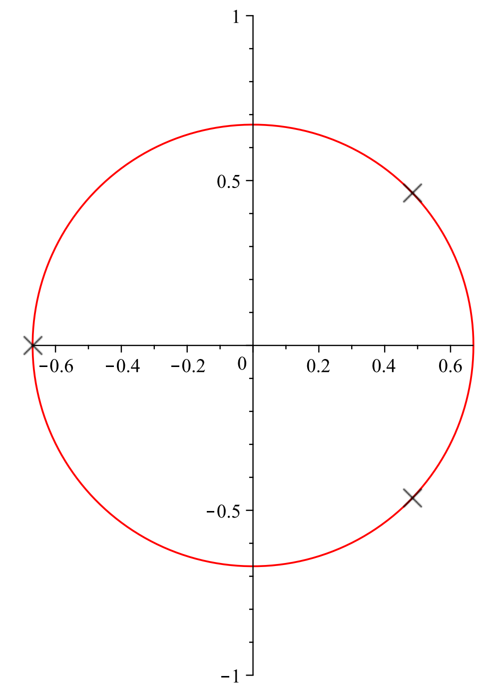


AMM, 2017

arXiv:1606.01131

Exp. Maths., 2019

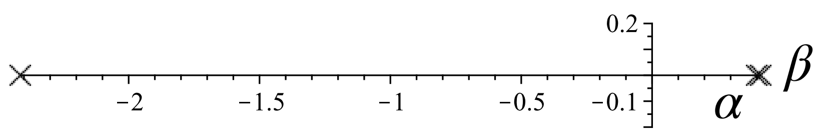
arXiv:1907.01232

Journées GDR EFI, June 2021

Joint work with Yann Bugeaud, Andrej Dujella, Wenjie Fang and Tomislav Pejković

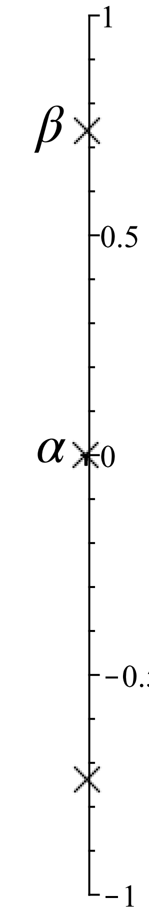
Close Roots of Polynomials

$$P \in \mathbb{Z}[X] \qquad P(\alpha) = P(\beta) = 0$$



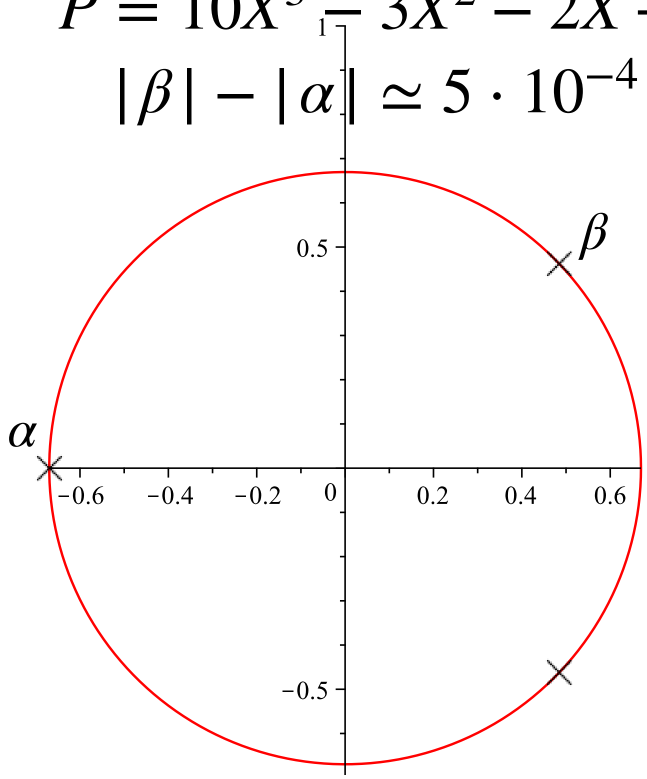
$$P = 5X^3 - 8X^2 - 9X + 2$$
$$\beta - \alpha \simeq 10^{-2}$$

$$P = 7X^3 + 5X^2 + 5X + 1$$
$$\Re\beta - \Re\alpha \simeq 6 \cdot 10^{-4}$$

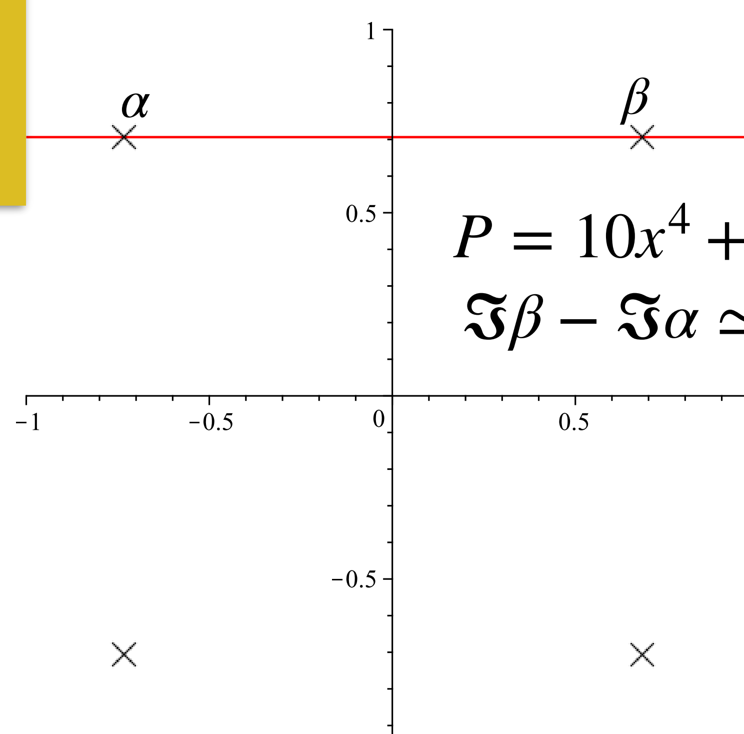


Aim: Bound precision needed to **decide** that two roots have identical value/real part/imaginary part/absolute value ?

$$P = 10X^3 - 3X^2 - 2X + 3$$
$$|\beta| - |\alpha| \simeq 5 \cdot 10^{-4}$$



$$P = 10x^4 + x^3 + 10$$
$$\Im\beta - \Im\alpha \simeq 6 \cdot 10^{-5}$$



Mahler's Bound

Def. Separation

$$\mathrm{sep}(P) := \min_{\substack{P(\alpha)=P(\beta)=0,\\ \alpha\neq\beta}} |\alpha-\beta|\,.$$

Def. Height

$$H\left(\sum_{i=0}^{d} a_i X^i\right) := \max_i |a_i|\,.$$

Thm. If $P \in \mathbb{Z}[X]$ has degree d,

$$\mathrm{sep}(P) > \kappa(d)\, H(P)^{-d+1}\,.$$

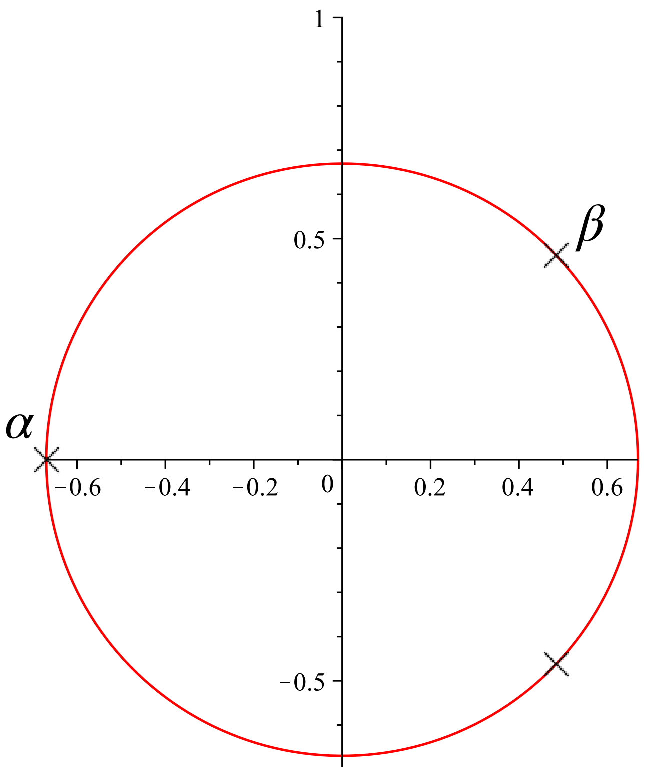
explicit function of d

not known to be tight (except for $d=3$)

worst known family gives $-(2d-1)/3$.

[Mahler 64; Evertse 04; BugeaudDujella 14]

Absolute Separation



Motivation: asymptotics of linear recurrences & diagonals

Def. Absolute Separation

$$\operatorname{abssep}(P) := \min_{\substack{P(\alpha)=P(\beta)=0,\\ |\alpha|\neq|\beta|}} \Big| |\alpha| - |\beta| \Big| .$$

Aims:

not the same as before

1. $\operatorname{abssep}(P) > \kappa(d)\, H(P)^{-e(d)}$ with $e(d)$ small;

2. families for small d with

$$\operatorname{abssep}(P_H) \underset{H\to\infty}{\sim} \kappa'\, H^{-e'} \text{ and } e'\,(\leq e) \text{ large}.$$

Note: Isolating disks of radius ε for all roots can be computed in time $\tilde{O}(d^3 + d^2 \log H(P) - d \log \varepsilon)$.

$e(d) \ll d^3$ would be nice

Results

Previously

$$e(d) \le d(d^2 + 2d - 1)/2 \qquad 1996$$

$$e(d) \le d^3/2 - d^2 - d/2 + 2 \qquad 2015$$

$$e(d) \le d^3/2 - d^2 - d/2 + 1 \quad (d \ge 4) \qquad 2019$$

New: $e(3) = 4, \quad 5 \le e(4) \le 12, \quad 6 \le e(5) \le 24, \quad 7 \le e(6) \le 30,$

$$e(d) \le (d-1)(d-2)(d-3)/2 = d^3/2 - 3d^2 + \cdots \qquad (d \ge 6)\,.$$

+ more precise bounds when one or two of the roots are real (→ 12,24)
+ bounds on the separation between real/imaginary parts

[GourdonSalvy 96; DubickasSha 15; Sha 19]

II. Proof Technique for Upper Bounds

Auxiliary Polynomials

From $P(X) = \sum_{i=0}^{d} a_i X^i = a_d \prod_{i=1}^{d} (X - \alpha_i) \in \mathbb{Z}[X]$ of height $H(P)$

construct

$$M(X) = a_d^{2(d-1)} \prod_{i<j} (X - (\alpha_i - \alpha_j)^2) \in \mathbb{Z}[X]$$

and lower bound its nonzero roots.

Prop. 1 [Cauchy] If $\alpha \neq 0$,

$$P(\alpha) = 0 \Rightarrow |\alpha| \geq \frac{1}{1 + H(P)}.$$

Prop. 2 [Symmetric fcns]

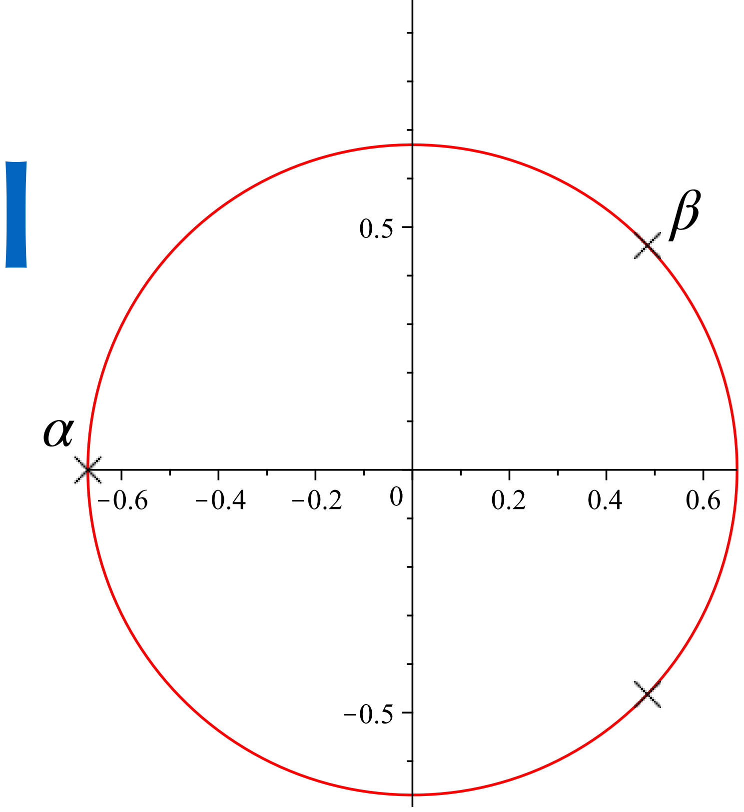
$G \in \mathbb{Z}[X_1, \ldots, X_d]$ symmetric with $\deg_{X_i} G \leq k$ for all i

$\Rightarrow a_d^k G(\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}[a_0, \ldots, a_d]$ of total degree $\leq k$.

Application to $M \rightarrow |\alpha_i - \alpha_j|^2 > \kappa H^{-2(d-1)}$.

Recovers Mahler's exponent

A Bigger Polynomial



$$a_d^{(d-1)(d-2)(d-3)} \prod_{\substack{i<j, \\ k<\ell, \\ \{i,j\}\cap\{k,\ell\}=\varnothing}} \left(X^{1/2} - (\alpha_i\alpha_j - \alpha_k\alpha_\ell) \right)$$

$$\Rightarrow (|\alpha|^2 - |\beta|^2)^2 \gg H^{-(d-1)(d-2)(d-3)}$$

gives exponent $(d-1)(d-2)(d-3)/2$ for the general case

More Auxiliary Polynomials

$$a_d^{2(d-1)} \prod_{i<j} (X - (\alpha_i + \alpha_j)^2)$$

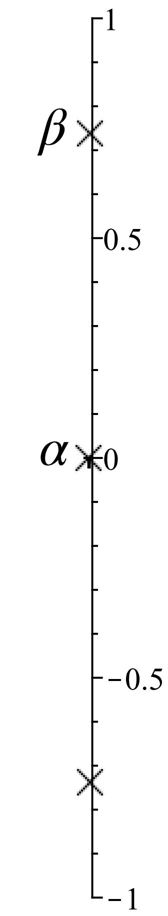
$$\alpha_j, \alpha_k \text{ real} \Rightarrow \big|\, |\alpha_j| - |\alpha_k| \,\big| > \kappa H^{-(d-1)}$$

optimal

$$a_d^{2(d-1)(d-2)} \prod_{i<j, k \notin \{i,j\}} (X - (\alpha_k^2 - \alpha_i \alpha_j))$$

$$\alpha_k \text{ real} \Rightarrow \big|\, |\alpha_j| - |\alpha_k| \,\big| > \kappa H^{-2(d-1)(d-2)}$$

Variants (×↦+)



$$a_d^{\frac{3}{2}(d-1)(d-2)} \prod_{i<j,k\notin\{i,j\}} \Big(X - (\alpha_i + \alpha_j - 2\alpha_k)\Big)$$

$$\alpha_k \text{ real} \Rightarrow \Big|\alpha_k - \Re\alpha_i\Big| > \kappa H^{-3(d-1)(d-2)/2}$$

$$a_d^{(d-1)(d-2)(d-3)} \prod_{\substack{i<j,\\ k<\ell,\\ \{i,j\}\cap\{k,\ell\}=\varnothing}} \Big(X^{1/2} - (\alpha_i + \alpha_j - \alpha_k - \alpha_\ell)\Big)$$

$$\Big|\Re\alpha_k - \Re\alpha_i\Big| > \kappa H^{-(d-1)(d-2)(d-3)/2}$$

and similarly for imaginary parts.

III. Experiments in Low Degree

Exhaustive Search

1. Solve the $(2H+1)^{d+1}$ pols in $\mathbb{Z}[X]_{\leq d}$ with height $\leq H$

and keep the records.

Ex.: $d = 3,\ H = 20 \rightarrow$ approx. 300,000 polynomials. (15 min.)
$d = 4,\ H = 20 \rightarrow$ approx. $115\ 10^6$ polynomials. (19 h)

2. Refine the search in the neighborhood of those;
look for patterns

	abssep	real root
$17x^3 + 9x^2 + 7x + 8$	$1.9\ 10^{-5}$	-0.7778352845
$102x^3 + 97x^2 + 71x + 40$	$1.5\ 10^{-8}$	-0.7319587393
$153x^3 - 97x^2 - 71x + 60$	$4.5\ 10^{-9}$	-0.7319587525
$181x^3 + 153x^2 + 112x + 71$	$9.0\ 10^{-10}$	-0.7320261422

$\approx 1 - \sqrt{3}$?

Degree 3: Optimal Exponent -4

Key polynomial:

$$P(X,Y) = X^3 - X^2 + 1 + \left(\frac{X^3}{2} - \frac{X^2}{3} + \frac{2}{3}X + 1\right)Y$$

Guessed from numerical coefficients

$$P(X,\sqrt{3}) = \frac{\sqrt{3}+2}{6}\left(X - \sqrt{3} + 1\right)\left(X^2 + aX + (\sqrt{3}-1)^2\right), \quad a < 2(\sqrt{3}-1).$$

discr<0 —> 3 roots with same modulus

Perturbation:

$P(X,\sqrt{3}+\epsilon)$ has a real root at $\sqrt{3} - 1 + (2-\sqrt{3})\epsilon + O(\epsilon^2)$

and a nonreal one with similar modulus, but a different O() term.

If p_n/q_n is the nth convergent of the continued fraction of $\sqrt{3}$,
$P_n(X) := 6q_nP(X, p_n/q_n) \in \mathbb{Z}[X], \quad \operatorname{abssep}(P_n) < \kappa H(P_n)^{-4}$.

Proof: $|p_n/q_n - \sqrt{3}| < 1/q_n^2$.

$$\sqrt{3} = 1 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{\ddots}}}$$

Perturbative Method (4≤deg≤6)

Principle

$$P(X,\epsilon) = R(X) + \epsilon\, Q(X)$$

$R(X)$: with roots of identical |.|

$Q(X)$: with undeterminate coefficients

1. Pick two nonconjugate roots α, β of R

2. Compute expansions $\alpha(\epsilon), \beta(\epsilon)$ of roots of P with $\alpha(0) = \alpha, \beta(0) = \beta$ in $\mathbb{Q}(\alpha)[q_0, \ldots, q_d][[\epsilon]]$ (or β)

3. Form the expansion of $|\alpha(\epsilon)|^2 - |\beta(\epsilon)|^2$ in $\mathbb{Q}(\alpha,\beta)[q_0, \ldots, q_d][[\epsilon]]$

4. Look for a nondegenerate integer solution of the system formed by its first coefficients

Demo

Results

deg	R	Q	exponent
4	$x^4 - 1$	$x^3 - x^2 + x - 5$	-5
4	$(x^2 - 1)(x^2 + x + 1)$	$x^3 - 3x - 4$	-5
6	$x^6 - 1$	$9x^5 - 9x^4 - 26x^3 - 9x^2 + 9x - 28$	-7
5	$(x^2 + ax + r^2)(x^2 + bx + r^2)$ $\lvert a \rvert < 2r, \lvert b \rvert < 2r$	Too big for the Gröbner basis computation	

Loop over small values of a, b, r gives:

$$a = -9,\ b = -11,\ r = 6,\ Q = X^5 - 213X^3 + 2404X^2 - 11088X + 20736$$

exponent -6

Conclusion: New ideas needed

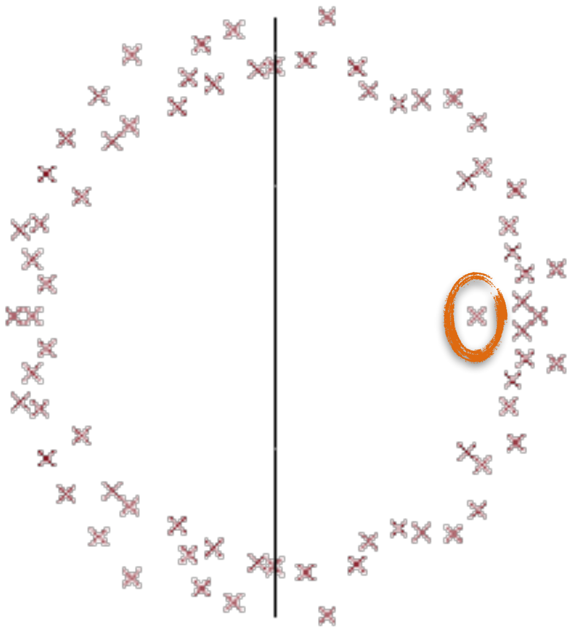
. to find polynomials with small absolute separation in low degree;

. to generalize the key polynomial of degree 3;

. to produce (or disprove the existence of) subcubic exponents;

. to obtain better bounds for the dominant roots?

The End



Conway's sequence