Absolute Root Separation

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Joint work with Yann Bugeaud, Andrej Dujella, Wenjie Fang and Tomislav Pejković

AMM, 2017
arXiv:1606.01131

Exp. Maths., 2019
arXiv:1907.01232
Close Roots of Polynomials

\[ P \in \mathbb{Z}[X] \quad P(\alpha) = P(\beta) = 0 \]

\[ P = 5X^3 - 8X^2 - 9X + 2 \]
\[ \beta - \alpha \simeq 10^{-2} \]

\[ P = 7X^3 + 5X^2 + 5X + 1 \]
\[ \Re\beta - \Re\alpha \simeq 6 \cdot 10^{-4} \]

\[ P = 10X^3 - 3X^2 - 2X + 3 \]
\[ |\beta| - |\alpha| \simeq 5 \cdot 10^{-4} \]

\[ P = 10x^4 + x^3 + 10 \]
\[ \Im\beta - \Im\alpha \simeq 6 \cdot 10^{-5} \]

**Aim:** Bound precision needed to decide that two roots have identical value/real part/imaginary part/absolute value?
Mahler’s Bound

**Def. Separation**

\[
\text{sep}(P) := \min_{P(\alpha) = P(\beta) = 0, \alpha \neq \beta} |\alpha - \beta|.
\]

**Def. Height**

\[
H\left(\sum_{i=0}^{d} a_i X^i\right) := \max_{i} |a_i|.
\]

**Thm.** If \( P \in \mathbb{Z}[X] \) has degree \( d \),

\[
\text{sep}(P) > \kappa(d) H(P)^{-d+1}.
\]

Explicit function of \( d \) not known to be tight (except for \( d = 3 \)) worst known family gives \(-(2d - 1)/3\).

[Mahler 64; Evertse 04; BugeaudDujella 14]
**Definition: Absolute Separation**

\[
\text{abssep}(P) := \min_{P(\alpha) = P(\beta) = 0, |\alpha| \neq |\beta|} \left| |\alpha| - |\beta| \right|.
\]

**Aims:**

1. \(\text{abssep}(P) > \kappa(d)H(P)^{-e(d)}\) with \(e(d)\) small;

2. families for small \(d\) with \(\text{abssep}(P_H) \sim \kappa' H^{-e'}\) and \(e'(\leq e)\) large.

**Note:** Isolating disks of radius \(\varepsilon\) for all roots can be computed in time \(\tilde{O}(d^3 + d^2 \log H(P) - d \log \varepsilon)\).

**Motivation:** asymptotics of linear recurrences & diagonals

\(e(d) \ll d^3\) would be nice
Results

Previously
\[
e(d) \leq d(d^2 + 2d - 1)/2 \quad 1996
\]
\[
e(d) \leq d^3/2 - d^2 - d/2 + 2 \quad 2015
\]
\[
e(d) \leq d^3/2 - d^2 - d/2 + 1 \quad (d \geq 4) \quad 2019
\]

New: \( e(3) = 4, \ 5 \leq e(4) \leq 12, \ 6 \leq e(5) \leq 24, \ 7 \leq e(6) \leq 30, \)
\[
e(d) \leq (d - 1)(d - 2)(d - 3)/2 = d^3/2 - 3d^2 + \cdots \quad (d \geq 6).
\]

+ more precise bounds when one or two of the roots are real (→ 12, 24)
+ bounds on the separation between real/imaginary parts

[GourdonSalvy 96; DubickasSha 15; Sha 19]
II. Proof Technique for Upper Bounds
Auxiliary Polynomials

From \( P(X) = \sum_{i=0}^{d} a_i X^i = a_d \prod_{i=1}^{d} (X - \alpha_i) \in \mathbb{Z}[X] \) of height \( H(P) \)

construct

\[
M(X) = a_d^{2(d-1)} \prod_{i<j} (X - (\alpha_i - \alpha_j)^2) \in \mathbb{Z}[X]
\]

and lower bound its nonzero roots.

**Prop. 1** [Cauchy] If \( \alpha \neq 0 \),

\[
P(\alpha) = 0 \Rightarrow |\alpha| \geq \frac{1}{1 + H(P)}.
\]

**Prop. 2** [Symmetric fcns] \( G \in \mathbb{Z}[X_1, \ldots, X_d] \) symmetric with \( \text{deg}_{X_i} G \leq k \) for all \( i \)

\[
\Rightarrow a_d^k G(\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}[a_0, \ldots, a_d] \text{ of total degree } \leq k.
\]

Application to \( M \rightarrow |\alpha_i - \alpha_j|^2 > \kappa H^{-(d-1)} \).
A Bigger Polynomial

\[ a_d^{(d-1)(d-2)(d-3)} \prod_{i < j, \ k < \ell, \ \{i, j\} \cap \{k, \ell\} = \emptyset} \left( X^{1/2} - (\alpha_i \alpha_j - \alpha_k \alpha_\ell) \right) \]

\[ \Rightarrow \left( |\alpha|^2 - |\beta|^2 \right)^2 \gg H^{-(d-1)(d-2)(d-3)} \]

gives exponent \( (d - 1)(d - 2)(d - 3)/2 \) for the general case.
More Auxiliary Polynomials

\[ a_d^{2(d-1)} \prod_{i<j} (X - (\alpha_i + \alpha_j)^2) \]

\[ a_d^{2(d-1)(d-2)} \prod_{i<j,k \notin \{i,j\}} (X - (\alpha_k^2 - \alpha_i \alpha_j)) \]

\[ \alpha_j, \alpha_k \text{ real} \Rightarrow \left| |\alpha_j| - |\alpha_k| \right| > \kappa H^{-(d-1)} \]

\[ \alpha_k \text{ real} \Rightarrow \left| |\alpha_j| - |\alpha_k| \right| > \kappa H^{-2(d-1)(d-2)} \]
Variants ($\times \mapsto +$)

$$a_d \frac{3}{2} (d-1)(d-2) \prod_{i<j,k \notin \{i,j\}} \left( X - (\alpha_i + \alpha_j - 2\alpha_k) \right)$$

$$\alpha_k \text{ real } \Rightarrow \left| \alpha_k - \Re \alpha_i \right| > \kappa H^{-(d-1)(d-2)/2}$$

$$a_d (d-1)(d-2)(d-3) \prod_{i<j,\ k<\ell,\ \{i,j\} \cap \{k,\ell\} = \emptyset} \left( X^{1/2} - (\alpha_i + \alpha_j - \alpha_k - \alpha_\ell) \right)$$

$$\left| \Re \alpha_k - \Re \alpha_i \right| > \kappa H^{-(d-1)(d-2)(d-3)/2}$$

and similarly for imaginary parts.
III. Experiments in Low Degree
Exhaustive Search

1. Solve the \((2H + 1)^{d+1}\) pols in \(\mathbb{Z}[X]_{\leq d}\) with height \(\leq H\)
   and keep the records.

Ex.: \(d = 3, \ H = 20 \rightarrow \text{approx. } 300,000\) polynomials. (15 min.)
\(d = 4, \ H = 20 \rightarrow \text{approx. } 115 \times 10^6\) polynomials. (19 h)

2. Refine the search in the neighborhood of those; look for patterns

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>abs sep</th>
<th>Real root</th>
</tr>
</thead>
<tbody>
<tr>
<td>(17x^3 + 9x^2 + 7x + 8)</td>
<td>(1.9 \times 10^{-5})</td>
<td>(-0.7778352845)</td>
</tr>
<tr>
<td>(102x^3 + 97x^2 + 71x + 40)</td>
<td>(1.5 \times 10^{-8})</td>
<td>(-0.7319587393)</td>
</tr>
<tr>
<td>(153x^3 - 97x^2 - 71x + 60)</td>
<td>(4.5 \times 10^{-9})</td>
<td>(-0.7319587525)</td>
</tr>
<tr>
<td>(181x^3 + 153x^2 + 112x + 71)</td>
<td>(9.0 \times 10^{-10})</td>
<td>(-0.7320261422)</td>
</tr>
</tbody>
</table>

\(\approx 1 - \sqrt{3}\)
Degree 3: Optimal Exponent -4

Key polynomial:

\[ P(X, Y) = X^3 - X^2 + 1 + \left( \frac{X^3}{2} - \frac{X^2}{3} + \frac{2}{3} X + 1 \right) Y \]

\[ P(X, \sqrt{3}) = \frac{\sqrt{3} + 2}{6} \left( X - \sqrt{3} + 1 \right) \left( X^2 + aX + (\sqrt{3} - 1)^2 \right), \quad a < 2(\sqrt{3} - 1). \]

Perturbation:

\[ P(X, \sqrt{3} + \epsilon) \] has a real root at \( \sqrt{3} - 1 + (2 - \sqrt{3})\epsilon + O(\epsilon^2) \)

and a nonreal one with similar modulus, but a different \( O() \) term.

If \( p_n/q_n \) is the \( n \)th convergent of the continued fraction of \( \sqrt{3} \),

\[ P_n(X) := 6q_n P(X, p_n/q_n) \in \mathbb{Z}[X], \quad \text{abssep}(P_n) < \kappa H(P_n)^{-4}. \]

Proof: \( |p_n/q_n - \sqrt{3}| < 1/q_n^2 \).
Perturbative Method (4 ≤ \text{deg} ≤ 6)

Principle

\[ P(X, \epsilon) = R(X) + \epsilon Q(X) \]

with roots of identical \( |.| \)

with undeterminate coefficients

1. Pick two nonconjugate roots \( \alpha, \beta \) of \( R \)

2. Compute expansions \( \alpha(\epsilon), \beta(\epsilon) \) of roots of \( P \)
   with \( \alpha(0) = \alpha, \beta(0) = \beta \)
   in \( \mathbb{Q}(\alpha)[q_0, \ldots, q_d][[\epsilon]] \)
   (or \( \beta \))

3. Form the expansion of \( |\alpha(\epsilon)|^2 - |\beta(\epsilon)|^2 \)
   in \( \mathbb{Q}(\alpha, \beta)[q_0, \ldots, q_d][[\epsilon]] \)

4. Look for a nondegenerate \textbf{integer} solution of the system
   formed by its first coefficients
## Results

<table>
<thead>
<tr>
<th>deg</th>
<th>$R$</th>
<th>$Q$</th>
<th>exponent</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$x^4 - 1$</td>
<td>$x^3 - x^2 + x - 5$</td>
<td>$-5$</td>
</tr>
<tr>
<td>4</td>
<td>$(x^2 - 1)(x^2 + x + 1)$</td>
<td>$x^3 - 3x - 4$</td>
<td>$-5$</td>
</tr>
<tr>
<td>6</td>
<td>$x^6 - 1$</td>
<td>$9x^5 - 9x^4 - 26x^3 - 9x^2 + 9x - 28$</td>
<td>$-7$</td>
</tr>
<tr>
<td>5</td>
<td>$(x^2 + ax + r^2)(x^2 + bx + r^2)$</td>
<td>$</td>
<td>a</td>
</tr>
</tbody>
</table>

Loop over small values of $a, b, r$ gives:

$a = -9, b = -11, r = 6, Q = X^5 - 213X^3 + 2404X^2 - 11088X + 20736$

exponent -6
Conclusion: New ideas needed

- to find polynomials with small absolute separation in low degree;
- to generalize the key polynomial of degree 3;
- to produce (or disprove the existence of) subcubic exponents;
- to obtain better bounds for the dominant roots?

The End