The differential Galois group of the rational function field

Annette Bachmayr
(joint work with David Harbater, Julia Hartmann, Michael Wibmer)

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Differential Galois theory

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key case: $K = \mathbb{C}$

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automorphism $\gamma \colon E \to E, e^x \mapsto ce^x$, $c \in K$

$$G = GL_1$$

The inverse differential Galois problem

Which linear algebraic groups over K are differential Galois groups over $K(x)$?

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Answer: All groups occur!

- proved for $K = \mathbb{C}$ in 1979 by Tretkoff/Tretkoff using analytic methods
- proved for arbitrary algebraically closed fields K in a series of papers by Kovacic (1969),..., Mitschi-Singer (1996),..., Hartmann (2002).

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$\pi\colon G = \{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a, b \in K, a \neq 0\} \twoheadrightarrow \mathrm{GL}_1,\ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto a,$

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$\rightsquigarrow$ Consider $\partial(y) = \begin{pmatrix} 1 & \frac{1}{x+1} \\ 0 & -1 \end{pmatrix} y$. Complete set of solutions:

$y_1 = \begin{pmatrix} e^x \\ 0 \end{pmatrix}, y_2 = \begin{pmatrix} g \\ e^{-x} \end{pmatrix},$

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Picard-Vessiot extension $E = F(e^x, g)$ with differential Galois group G and restriction $\mathrm{Aut}^\partial(E/F) \twoheadrightarrow \mathrm{Aut}^\partial(L/F)$ corresponds to π.

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Picard-Vessiot extensions of infinite type

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$$\partial(y) = A_i y, i \in I$$

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$\rightsquigarrow$ The differential Galois group of E/F is again defined as $G = \mathrm{Aut}^{\partial}(E/F)$

Fact: G is a proalgebraic group over K, i.e., it is an inverse limit $\varprojlim G_i$ of linear algebraic groups over K, or, equivalently, an affine group scheme over K.

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Note: $\mathrm{Aut}^{\partial}(\tilde{F}/F) = \varprojlim_{E \subseteq \tilde{E}} \mathrm{Aut}^{\partial}(E/F)$ is a proalgebraic group

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4. *Instead of only asking which algebraic groups occur as Galois groups one can ask which proalgebraic groups occur:* Is every proalgebraic group with $\mathrm{rank}(G) \leq |K|$ the differential Galois group of some Picard-Vessiot extension (of infinite type)?

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5. *Combine 2 and 4:* Given *proalgebraic* groups $\pi\colon G \twoheadrightarrow H$ and L/F Picard-Vessiot extension (of infinite type) with differential Galois group H, is there a solution to this differential embedding problem if we assume $\operatorname{rank}(G) \leq |K|$ and $\operatorname{rank}(H) < |K|$?

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Let K be an algebraically closed field of characteristic zero and infinite transcendence degree over $\mathbb{Q}$.

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- *Then we can give affirmative answers to all these five generalizations of the inverse differential Galois problem.*
- *Moreover, we show that absolute differential Galois group of $K(x)$ is free of rank $|K|$.*

Matzat's conjecture

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Let K be an algebraically closed field of characteristic zero. Then the absolute differential Galois group of $K(x)$ is free of rank $|K|$.

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The conjecture was inspired by

Theorem (Geometric Shafarevich conjecture)

Let K be an algebraically closed field. Then the absolute Galois group of $K(x)$ is free of rank $|K|$.

Solved in 1964 by Douady for the case $\mathrm{char}(K) = 0$ and for general K in 1995 by Harbater and Pop.

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Proposition

Matzat's conjecture holds if and only

(i) *every differential embedding problem $(G \twoheadrightarrow H, L)$ over $K(x)$ with G, H of finite type has a solution and*

(ii) *every split admissible differential embedding problem over F has a solution.*

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Here, a differential embedding problem $(G \twoheadrightarrow H, L)$ with proalgebraic groups G, H and $L/K(x)$ a Picard-Vessiot extension (of infinite type) with Galois group H is called

- *admissible*, if $\mathrm{rank}(H) < |K|$ and $N = \ker(G \twoheadrightarrow H)$ is of finite type
- *split*, if $G \twoheadrightarrow H$ splits, i.e. $G = N \rtimes H$

Solving a special case over $K(x)$

Consider $F = K(x)$ with K algebraically closed of infinite transcendence degree. Let $G = N \rtimes H$ be an algebraic group and let L/F be a Picard-Vessiot extension with Galois group H.

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Previous result (joint with David Harbater, Julia Hartmann and Florian Pop): Even in the more general situation of large fields K of infinite transcendence degree, such an embedding problem can be solved.

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Definition

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Example:

- PAC fields, in particular algebraically closed fields
- K complete wrt non-trivial absolute value, e.g. $\mathbb{R}$, $\mathbb{Q}_p$, $k((t))$
- fraction fields of domains that are Henselian wrt non-trivial ideal, e.g. $K = k((t_1, \dots, t_n))$, Puiseaux series fields,...

From algebraic groups to proalgebraic groups

Consider $F = K(x)$. Let $G = N \rtimes H$ be a proalgebraic group over K and L/F a Picard-Vessiot extension with Galois group H such that the embedding problem $(N \rtimes H,\ L)$ is admissible, in particular, N is of finite type.

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Write $H = \varprojlim_{i\in I} H_i$ with $H_i = H/U_i$ of finite type and $L = \varinjlim_{i\in I} L_i$ with L_i/F Picard-Vessiot with Galois group H_i.

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Use that N is of finite type $\rightsquigarrow$ U_i acts trivially on N "for sufficiently many $i \in I$", i.e., $\exists J \subseteq I$ s.t. U_j acts trivially on N for every $j \in J$ and $\bigcap_{j \in J} U_j = 1$.

Then $H = \varprojlim_{j \in J} H_j$, $G = \varprojlim_{j \in J} N \rtimes H_j$ and $L = \varinjlim_{j \in J} L_j$. Hence for each j, $(N \rtimes H_j,\ L_j)$ defines a differential embedding problem (of finite type) and we can obtain solutions E_j for every j.

$\rightsquigarrow$ show that these solutions are compatible, i.e., $\varinjlim_{j \in J} E_j$ yields a solution to the given differential embedding problem $(N \rtimes H, L)$.

References

[1] *Free differential Galois groups*, with David Harbater, Julia Hartmann and Michael Wibmer, **Transactions of the American Mathematical Society (to appear)**.

[2] *The differential Galois group of the rational function field*, with David Harbater, Julia Hartmann and Michael Wibmer, **Advances in Mathematics (to appear)**.

Notes

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Let X be a set. A proalgebraic group Γ together with a map $\iota\colon X \to \Gamma(\bar{K})$ such that "ι converges to 1" is called the free proalgebraic group on X if for all other such pairs (Γ', ι') there exists a unique morphism $\psi\colon \Gamma \to \Gamma'$ such that

$$\begin{array}{ccc} X & \xrightarrow{\ \iota\ } & \Gamma(\bar{K}) \\ & {\scriptstyle \iota'} \searrow \quad \swarrow {\scriptstyle \psi_{\bar{K}}} & \\ & \Gamma'(\bar{K}) & \end{array}$$

commutes.