Presenting isomorphic finitely presented modules by equivalent matrices: a constructive approach

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Linear systems and finitely presented left D-modules

◇ D ring of functional operators, $R \in D^{q \times p}$, and $\mathcal{F}$ a left D-module

◇ Consider the linear system $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$.

◇ We associate the following finitely presented left D-module:

$$M = D^{1\times p}/(D^{1\times q}\,R),$$

given by the finite presentation:

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0, \\ \lambda = (\lambda_1, \ldots, \lambda_q) & \longmapsto & \lambda\, R & & & & \end{array}$$

◇ Malgrange's isomorphism: $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$

◇ Algebraic analysis: the linear system $\ker_{\mathcal{F}}(R.)$ can be studied by means of the finitely presented left D-module M.

Example in the theory of linear elasticity (*Pommaret'01*)

$\diamond$ $(S)\left\{\begin{array}{l}\partial_1\,\xi_1=0,\\ \frac{1}{2}\,(\partial_2\,\xi_1+\partial_1\,\xi_2)=0,\\ \partial_2\,\xi_2=0.\end{array}\right.$ $\quad(S')\left\{\begin{array}{ll}\partial_1\,\zeta_1=0, & \partial_2\,\zeta_1-\zeta_2=0,\\ \partial_1\,\zeta_2=0, & \partial_1\,\zeta_3+\zeta_2=0,\\ \partial_2\,\zeta_3=0, & \partial_2\,\zeta_2=0.\end{array}\right.$

$\diamond$ We consider the ring $D=\mathbb{Q}[\partial_1,\partial_2]$ and we have

$$(S)\Leftrightarrow\overbrace{\begin{pmatrix}\partial_1 & 0\\ \frac{1}{2}\,\partial_2 & \frac{1}{2}\,\partial_1\\ 0 & \partial_2\end{pmatrix}}^{R}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=0,\ (S')\Leftrightarrow\overbrace{\begin{pmatrix}\partial_1 & 0 & 0\\ \partial_2 & -1 & 0\\ 0 & \partial_1 & 0\\ 0 & 1 & \partial_1\\ 0 & 0 & \partial_2\\ 0 & \partial_2 & 0\end{pmatrix}}^{R'}\begin{pmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{pmatrix}=0.$$

$\diamond$ We associate $M=D^{1\times 2}/(D^{1\times 3}\,R)$ and $M'=D^{1\times 3}/(D^{1\times 6}\,R')$.

Two theorems about isomorphisms and equivalences

Theorem (Fitting, 1936)

Two matrices $R \in D^{q \times p}$ and $R' \in D^{q' \times p'}$ presenting isomorphic left D-modules can be inflated with blocks of 0 and I to get equivalent matrices presenting the same left D-modules.

Theorem (Warfield, 1978)

If two positive integers s and r satisfy

$$s \leq \min(p+q', q+p'), \quad \mathrm{sr}(D) \leq \max(p+q'-s, q+p'-s),$$
$$r \leq \min(p, p'), \quad \mathrm{sr}(D) \leq \max(p-r, p'-r),$$

then we can remove s blocks of zeros and r blocks of identity.

◇ Goal of the talk: give constructive versions of both theorems

Example in the theory of linear elasticity

◇ We will prove that $M = D^{1\times 2}/(D^{1\times 3}\, R) \cong M' = D^{1\times 3}/(D^{1\times 6}\, R')$.

◇ Constructive version of Fitting's theorem: compute $X \in \mathrm{GL}_5(D)$ and $Y \in \mathrm{GL}_{14}(D)$ such that:

$$\left(\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & \partial_1 & 0 & 0 \\ 0 & 0 & \partial_2 & -1 & 0 \\ 0 & 0 & 0 & \partial_1 & 0 \\ 0 & 0 & 0 & 1 & \partial_1 \\ 0 & 0 & 0 & 0 & \partial_2 \\ 0 & 0 & 0 & \partial_2 & 0 \end{array}\right) = Y^{-1} \left(\begin{array}{cc|ccc} \partial_1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\,\partial_2 & \frac{1}{2}\,\partial_1 & 0 & 0 & 0 \\ 0 & \partial_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) X.$$

Example in the theory of linear elasticity

$\diamond$ Constructive version of Warfield's theorem (slight generalization): compute $X' \in \mathrm{GL}_3(D)$ and $Y' \in \mathrm{GL}_7(D)$ such that:

$$\left(\begin{array}{ccc} 0 & 0 & 0 \\ \hline \partial_1 & 0 & 0 \\ \partial_2 & -1 & 0 \\ 0 & \partial_1 & 0 \\ 0 & 1 & \partial_1 \\ 0 & 0 & \partial_2 \\ 0 & \partial_2 & 0 \end{array}\right) = Y'^{-1} \left(\begin{array}{cc|c} \partial_1 & 0 & 0 \\ \frac{1}{2}\,\partial_2 & \frac{1}{2}\,\partial_1 & 0 \\ 0 & \partial_2 & 0 \\ \hline 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) X'.$$

Part I

Definition of isomorphic finitely presented left D-modules in terms of matrix equalities

Homomorphisms in terms of matrix equality

⋄ Let D be a ring of functional operators.

⋄ Let $R \in D^{q \times p}$, $\quad R' \in D^{q' \times p'}$ be two matrices.

⋄ We have the following commutative exact diagram:

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\ \downarrow \; .Q & & \downarrow \; .P & & \downarrow \; f & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow 0. \end{array}$$

$$\exists \, f : M \to M' \Longleftrightarrow \exists \, P \in D^{p \times p'}, \; Q \in D^{q \times q'} \text{ such that:}$$

$$R \, P = Q \, R'.$$

Moreover, we have $f(\pi(\lambda)) = \pi'(\lambda \, P)$, for all $\lambda \in D^{1 \times p}$.

Isomorphisms in terms of matrix equalities

◇ $f \in \hom_D(M, M')$ given by P and Q such that $R\,P = Q\,R'$.

◇ $f \in \mathrm{iso}_D(M, M')$ if and only if there exist $P' \in D^{p' \times p}$, $Q' \in D^{q' \times q}$, $Z \in D^{p \times q}$ and $Z' \in D^{p' \times q'}$ satisfying the following relations:

$$R'\,P' = Q'\,R, \quad P\,P' + Z\,R = I_p, \quad P'\,P + Z'\,R' = I_{p'}.$$

Then, there exist $Z_2 \in D^{q \times r}$ and $Z_2' \in D^{q' \times r'}$ such that:

$$Q\,Q' + R\,Z + Z_2\,R_2 = I_q, \quad Q'\,Q + R'\,Z' + Z_2'\,R_2' = I_{q'},$$

where $\ker_D(.R) = D^{1 \times r}\,R_2$ and $\ker_D(.R') = D^{1 \times r'}\,R_2'$.

Implementations

◇ All the matrices appearing in the relations defining (iso)morphisms can be computed from R and R' using (non-commutative) Gröbner basis calculations.

◇ Implementations:

1. `Maple` package OREMORPHISMS (C.Q.) based on OREMODULES (*Chyzak, Q., Robertz*);
2. `Mathematica` package OREALGEBRAICANALYSIS (*C., Q., Tõnso*) based on HOLONOMICFUNCTIONS (*Koutschan*);
3. `CapAndHomalg` (*Barakat et al*).

Example in the theory of linear elasticity

◇ We have $R\,P = Q\,R'$ with:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}.$$

◇ We have $R'\,P' = Q'\,R$, $P\,P' + Z\,R = I_p$, and $P'\,P + Z'\,R' = I_{p'}$ with:

$$P' = \begin{pmatrix} 1 & 0 \\ \partial_2 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \partial_2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 2\,\partial_2 & -\partial_1 \end{pmatrix},$$

$$Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

◇ This proves that $M = D^{1\times 2}/(D^{1\times 3}\,R) \cong M' = D^{1\times 3}/(D^{1\times 6}\,R')$.

Part II

Constructive version of Fitting's theorem

Statement of the problem

◇ Let $M = D^{1\times p}/(D^{1\times q}\, R)$ and $M' = D^{1\times p'}/(D^{1\times q'}\, R')$.

◇ Assume that $M \cong M'$

Theorem (Fitting, 1936)

Two matrices $R \in D^{q\times p}$ and $R' \in D^{q'\times p'}$ presenting isomorphic left D-modules can be inflated with blocks of 0 and I to get equivalent matrices presenting the same left D-modules.

◇ Goal: Compute inflations L and L' of R and R' with blocks of 0 and I and two unimodular matrices X and Y such that:

- L and L' respectively define a finite presentation of M and M';
- L and L' are equivalent matrices, i.e., we have $L' = Y^{-1}\, L\, X$.

◇ We give explicit formulas for X and Y in terms of the matrices defining the isomorphism.

Explicit Fitting's theorem

◇ Let $\left\{ \begin{array}{l} n = q + p' + p + q' \\ m = p + p' \end{array} \right.$ and define the matrices:

1.
$$X = \begin{pmatrix} I_p & P \\ -P' & I_{p'} - P'\,P \end{pmatrix} \in \mathrm{GL}_m(D), \quad X^{-1} = \begin{pmatrix} I_p - P\,P' & -P \\ P' & I_{p'} \end{pmatrix}.$$

2.
$$Y = \begin{pmatrix} I_q & 0 & R & Q \\ 0 & I_{p'} & -P' & Z' \\ -Z & P & 0 & P\,Z' - Z\,Q \\ -Q' & -R' & 0 & Z_2'\,R_2' \end{pmatrix} \in \mathrm{GL}_n(D),$$

with inverse given by

$$Y^{-1} = \begin{pmatrix} Z_2\,R_2 & 0 & -R & -Q \\ P'\,Z - Z'\,Q' & 0 & P' & -Z' \\ Z & -P & I_p & 0 \\ Q' & R' & 0 & I_{q'} \end{pmatrix}.$$

Explicit Fitting's theorem

◇ Let $L = \begin{pmatrix} R & 0 \\ 0 & I_{p'} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in D^{n \times m}$, $L' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I_p & 0 \\ 0 & R' \end{pmatrix} \in D^{n \times m}$.

◇ The following commutative exact diagram holds:

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ D^{1\times n} & \xrightarrow{.L} & D^{1\times m} & \xrightarrow{\pi \oplus 0_{p'}} & M & \longrightarrow & 0 \\ \downarrow .Y & & \downarrow .X & & \downarrow f & & \\ D^{1\times n} & \xrightarrow{.L'} & D^{1\times m} & \xrightarrow{0_p \oplus \pi'} & M' & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & & 0 & & 0 & & \end{array}$$

and L and L' are equivalent matrices, i.e.,

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I_p & 0 \\ 0 & R' \end{pmatrix} = Y^{-1} \begin{pmatrix} R & 0 \\ 0 & I_{p'} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} X.$$

Example in the theory of 2D linear elasticity

◇ We get

$$\left(\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & \partial_1 & 0 & 0 \\ 0 & 0 & \partial_2 & -1 & 0 \\ 0 & 0 & 0 & \partial_1 & 0 \\ 0 & 0 & 0 & 1 & \partial_1 \\ 0 & 0 & 0 & 0 & \partial_2 \\ 0 & 0 & 0 & \partial_2 & 0 \end{array}\right) = Y^{-1} \left(\begin{array}{cc|ccc} \partial_1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\,\partial_2 & \frac{1}{2}\,\partial_1 & 0 & 0 & 0 \\ 0 & \partial_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) X.$$

Example in the theory of 2D linear elasticity

$\diamond$ $X \in \mathrm{GL}_5(D)$ and $Y \in \mathrm{GL}_{14}(D)$ are given by:

$$X = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ -\partial_2 & 0 & -\partial_2 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$Y = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \partial_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2}\,\partial_2 & \frac{1}{2}\,\partial_1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \partial_2 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\partial_2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -\partial_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\partial_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\partial_2 & 0 & 0 & 0 & -\partial_1 & 0 & 0 & 0 & -\partial_2 & \partial_1 & 1 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & -1 & -\partial_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -\partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2\,\partial_2 & \partial_1 & 0 & -\partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_2 & \partial_1 & 1
\end{pmatrix}.$$

Part III

Constructive version of Warfield's theorem

Statement of the problem

◇ Let $M = D^{1\times p}/(D^{1\times q}\, R) \cong M' = D^{1\times p'}/(D^{1\times q'}\, R')$.

◇ Assume that $L' = Y^{-1}\, L\, X$ as before (Fitting's theorem).

Theorem (Warfield, 1978)

If two positive integers s and r satisfy

$$\begin{array}{ll} s \leq \min(p+q', q+p'), & \mathrm{sr}(D) \leq \max(p+q'-s, q+p'-s), \\ r \leq \min(p, p'), & \mathrm{sr}(D) \leq \max(p-r, p'-r), \end{array}$$

then we can remove s blocks of zeros and r blocks of identity.

◇ Goal: Compute $X_r \in \mathrm{GL}_{m-r}(D)$ and $Y_{s,r} \in \mathrm{GL}_{n-s-r}(D)$ such that:

$$\begin{pmatrix} 0 & 0 \\ I_{p-r} & 0 \\ 0 & R' \end{pmatrix} = Y_{s,r}^{-1} \begin{pmatrix} R & 0 \\ 0 & I_{p'-r} \\ 0 & 0 \end{pmatrix} X_r.$$

Stable rank of a ring D

Definitions (e.g., McConnell & Robson)

1. $u \in D^n$ is unimodular if $\exists v \in D^{1\times n}$ such that $v\, u = 1$.
2. $u = (u_1, \ldots, u_n)^T \in D^n$ is stable if $\exists d_1, \ldots, d_{n-1} \in D$ such that $(u_1 + d_1\, u_n, \ldots, u_{n-1} + d_{n-1}\, u_n)^T \in D^{n-1}$ is unimodular.
3. An integer r is said to be in the stable range of D if $\forall n > r$, a unimodular vector $u \in D^n$ is stable.
4. The stable rank $\mathrm{sr}(D)$ of D is the smallest positive integer that is in the stable range of D. If no such integer exists, then $\mathrm{sr}(D) = +\infty$.

◇ Examples:

1. If D is a principal domain, then $\mathrm{sr}(D) \geq 2$;
2. $\forall n \geq 1$, we have $\mathrm{sr}(\mathbb{Q}[x_1, \ldots, x_n]) = n + 1$ (*Vasershtein'71*);
3. $\mathrm{sr}(A_n(k)) = 2$ and $\mathrm{sr}(B_n(k)) = 2$ (Stafford's theorem - *Stafford'78*).

A key result based on $\mathrm{sr}(D)$

Lemma

Let D be a ring and n, m two integers such that $\mathrm{sr}(D) \leq m$.
Let $u \in D^{n+m+1}$ be a unimodular column vector such that we have:

$$v\,u = \begin{pmatrix} v_n & v_{m+1} \end{pmatrix} \begin{pmatrix} u_n \\ u_m \\ u_1 \end{pmatrix} = 1,$$

where $v_n \in D^{1\times n}$, $v_{m+1} \in D^{1\times(m+1)}$, $u_n \in D^n$, $u_m \in D^m$, $u_1 \in D$.
Then, there exist $c_1 \in D$, $\tilde{u}_m \in D^m$, $\tilde{v}_m \in D^{1\times m}$ such that we have:

$$\begin{pmatrix} c_1\,v_n & \tilde{v}_m \end{pmatrix} \begin{pmatrix} u_n \\ u_m + \tilde{u}_m\,u_1 \end{pmatrix} = 1.$$

◇ No general algorithm for computing c_1, $\tilde{u}_m$, and $\tilde{v}_m$ in any ring D.

◇ But algorithms and heuristics are implemented for some particular rings D (as for instance Weyl algebras) and allow to treat examples.

An iterative process

◇ Starting point: $X \in \mathrm{GL}_m(D)$ and $Y \in \mathrm{GL}_n(D)$ such that $L' = Y^{-1} L X$.

◇ We assume that s and r satisfy the hypothesis of Warfield's theorem.

◇ Remove s zeros rows: From $Y_0 := Y$, we compute recursively matrices $Y_1, \ldots, Y_s$ such that: $\forall i = 1, \ldots, s$,

$$Y_i \in \mathrm{GL}_{n-i}(D), \quad \underbrace{\begin{pmatrix} 0 & 0 \\ I_p & 0 \\ 0 & R' \end{pmatrix}}_{L'_i \in D^{(n-i) \times m}} = Y_i^{-1} \underbrace{\begin{pmatrix} R & 0 \\ 0 & I_{p'} \\ 0 & 0 \end{pmatrix}}_{L_i \in D^{(n-i) \times m}} X.$$

◇ Remove r identity blocks: From $Y_{s,0} := Y_s$ and $X_0 := X$, we compute recursively $Y_{s,1}, \ldots, Y_{s,r}$ and $X_1, \ldots, X_r$ such that: $\forall j = 1, \ldots, r$,

$$Y_{s,j} \in \mathrm{GL}_{n-s-j}(D),\ X_j \in \mathrm{GL}_{m-j}(D), \quad \underbrace{\begin{pmatrix} 0 & 0 \\ I_{p-j} & 0 \\ 0 & R' \end{pmatrix}}_{L'_{s,j} \in D^{(n-s-j) \times (m-j)}} = Y_{s,j}^{-1} \underbrace{\begin{pmatrix} R & 0 \\ 0 & I_{p'-j} \\ 0 & 0 \end{pmatrix}}_{L_{s,j} \in D^{(n-s-j) \times (m-j)}} X_j.$$

Procedure to remove zero rows: general description

◇ Hyp.: we have computed $Y_{i-1} \in \mathrm{GL}_{n-i+1}(D)$ s.t. $L'_{i-1} = Y_{i-1}^{-1} \, L_{i-1} \, X$.

$$\begin{array}{ccc}
D^{1\times(n-i)} & \xrightarrow{.L'_i} & D^{1\times m} \\
\uparrow .G'_i & & \uparrow .I_m \\
D^{1\times(n-i+1)} & \xrightarrow{.L'_{i-1}} & D^{1\times m} \\
\uparrow .Y_{i-1} & & \uparrow .X \\
D^{1\times(n-i+1)} & \xrightarrow{.L_{i-1}} & D^{1\times m} \\
\uparrow .W_i & & \uparrow .I_m \\
D^{1\times(n-i+1)} & \xrightarrow{.L_{i-1}} & D^{1\times m} \\
\uparrow .H_i & & \uparrow .I_m \\
D^{1\times(n-i)} & \xrightarrow{.L_i} & D^{1\times m}
\end{array}$$

$$\Longrightarrow L_i \, X = \underbrace{(H_i \, W_i \, Y_{i-1} \, G'_i)}_{Y_i \in \mathrm{GL}_{n-i}(D)} \, L_{i-1}, \quad Y_i^{-1} = H'_i \, Y_{i-1}^{-1} \, W_i^{-1} \, G_i.$$

Procedure to remove zero rows: matrix computations

◇ We decompose Y_{i-1} and its inverse Y_{i-1}^{-1} by blocks as follows:

$$Y_{i-1} = \begin{pmatrix} Y_{11} \\ Y_{21} \\ Y_{31} \end{pmatrix} \begin{matrix} \leftarrow q+p' \\ \leftarrow p+q'-i \\ \leftarrow 1 \end{matrix} \qquad Y_{i-1}^{-1} = \begin{pmatrix} \underset{\downarrow}{\overset{q+p'}{}} Y'_{11} & Y'_{12} & \underset{\downarrow}{\overset{1}{}} Y'_{13} \end{pmatrix}, \quad Y'_{12} \text{ has } p+q'-i \text{ columns } (\uparrow)$$

◇ As $\mathrm{sr}(D) \leq p + q' - s$, the key lemma implies that there exist $c \in D$, $u \in D^{p+q'-i}$, $v \in D^{1\times(p+q'-i)}$ such that, if $k = q + p' - (i-1)$, then

$$\begin{pmatrix} c\,(Y'_{11})_{k.} & v \end{pmatrix} \begin{pmatrix} (Y_{11})_{.k} \\ (Y_{21})_{.k} + u\,(Y_{31})_{k} \end{pmatrix} = 1.$$

◇ Assume $c \in D$, $u \in D^{p+q'-i}$, $v \in D^{1\times(p+q'-i)}$ have been computed.

Procedure to remove zero rows: matrix computations

◇ $W_i = \begin{pmatrix} I_{q+p'} & 0 & 0 \\ 0 & I_{p+q'-i} & u \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_{n-i+1}(D) \rightsquigarrow$ commuting square diagram:

$$\begin{array}{ccc} D^{1\times(n-i+1)} & \xrightarrow{.L_{i-1}} & D^{1\times m} \\ .W_i^{-1} \downarrow\uparrow .W_i & & .I_m \downarrow\uparrow .I_m \\ D^{1\times(n-i+1)} & \xrightarrow{.L_{i-1}} & D^{1\times m} \end{array}$$

◇ Let $\tilde{\ell}_i = \begin{pmatrix} c\,(Y'_{11})_{k\cdot} & v \end{pmatrix}$, $\ell_i = \begin{pmatrix} c\,(Y'_{11})_{k\cdot} & v & 0 \end{pmatrix}$, $F_i = \begin{pmatrix} Y_{11} \\ Y_{21} + u\,Y_{31} \end{pmatrix}$.

Procedure to remove zero rows: matrix computations

$\diamond$ $G_i = \begin{pmatrix} I_{n-i} - (F_i)_{.k}\,\tilde{\ell}_i \\ \tilde{\ell}_i \end{pmatrix}$, $H_i = \begin{pmatrix} I_{n-i} - (F_i)_{.k}\,\tilde{\ell}_i & (F_i)_{.k} \end{pmatrix}$ satisfy $H_i\, G_i = I_{n-i}$ and the following square diagrams commutes:

$$\begin{array}{ccc} D^{1\times(n-i+1)} & \xrightarrow{.L_{i-1}} & D^{1\times m} \\ \uparrow .H_i & & \uparrow .I_m \\ D^{1\times(n-i)} & \xrightarrow{.L_i} & D^{1\times m} \end{array}$$

$\diamond$ $G_i' = \left(I_{n-i+1} - (f_k^{n-i+1})^T\, \ell_i\, W_i\, Y_{i-1}\right) \begin{pmatrix} I_{k-1} & 0 \\ 0 & 0 \\ 0 & I_{p+q'} \end{pmatrix}$, $H_i' = \begin{pmatrix} I_{k-1} & 0 & 0 \\ 0 & 0 & I_{p+q'} \end{pmatrix}$,

satisfy $H_i'\, G_i' = I_{n-i}$ and the following square diagrams commutes:

$$\begin{array}{ccc} D^{1\times(n-i)} & \xrightarrow{.L_i'} & D^{1\times m} \\ \uparrow .G_i' & & \uparrow .I_m \\ D^{1\times(n-i+1)} & \xrightarrow{.L_{i-1}'} & D^{1\times m} \end{array}$$

Procedure to remove zero rows: matrix computations

◇ $Y_i = H_i\, W_i\, Y_{i-1}\, G_i' \in \mathrm{GL}_{n-i}(D)$ with inverse $Y_i^{-1} = H_i'\, Y_{i-1}^{-1}\, W_i^{-1}\, G_i$.

◇ The following commutative exact diagram holds

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
D^{1\times(n-i)} & \xrightarrow{.L_i} & D^{1\times m} & \xrightarrow{\pi \oplus 0_{p'}} & M & \longrightarrow & 0 \\
\downarrow .Y_i & & \downarrow .X & & \downarrow f & & \\
D^{1\times(n-i)} & \xrightarrow{.L_i'} & D^{1\times m} & \xrightarrow{0_p \oplus \pi'} & M' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

and L_i and L_i' are equivalent matrices.

◇ Matrices given explicitly in terms of those of the previous equivalence.

Problem reduced to computing $c \in D$, $u \in D^{p+q'-i}$, $v \in D^{1\times(p+q'-i)}$ s.t.:

$$\left(c\,(Y_{11}')_{k.} \quad v\right) \begin{pmatrix} (Y_{11})_{.k} \\ (Y_{21})_{.k} + u\,(Y_{31})_k \end{pmatrix} = 1.$$

Example in the theory of 2D linear elasticity

◇ We have $q = 3$, $p = 2$, $q' = 6$, and $p' = 3$ so that $n = 14$ and $m = 5$.

◇ We have $q + p' = 6 \leq p + q' = 8$, and $\mathrm{sr}(\mathbb{Q}[\partial_1, \partial_2]) = 3$.

◇ The integer s has to satisfy:

$$s \leq \min(p + q', q + p') = 6, \ \mathrm{sr}(D) \leq \max(p + q' - s, q + p' - s) = 8 - s,$$

$\Longrightarrow$ We can remove $s = 5$ zero rows.

Example in the theory of 2D linear elasticity

◇ To remove the first zero row, we are reduced to solving:

$$\begin{pmatrix} c & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & v \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -\partial_1 \\ -\partial_2 \end{pmatrix} + 0\, u \end{pmatrix} = 1,$$

◇ A solution is given by $c = 0$, $v = (0\ 1\ 0\ 0\ 0\ 0\ 0)$, and $u = (0\ 0\ 0\ 0\ 0\ 0\ 0)^T$

Example in the theory of 2D linear elasticity

◇ From the above formulas, we get $Y_1 \in \mathrm{GL}_{13}(D)$ s.t. $L_1' = Y_1^{-1} L_1' X$.

◇ Applying recursion, we compute $Y_i \in \mathrm{GL}_{14-i}(D)$, $i = 2, \ldots, 5$ so that we finally get:

$$\underbrace{\left(\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & \partial_1 & 0 & 0 \\ 0 & 0 & \partial_2 & -1 & 0 \\ 0 & 0 & 0 & \partial_1 & 0 \\ 0 & 0 & 0 & 1 & \partial_1 \\ 0 & 0 & 0 & 0 & \partial_2 \\ 0 & 0 & 0 & \partial_2 & 0 \end{array}\right)}_{L_5'} = Y_5^{-1} \underbrace{\left(\begin{array}{cc|ccc} \partial_1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\,\partial_2 & \frac{1}{2}\,\partial_1 & 0 & 0 & 0 \\ 0 & \partial_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)}_{L_5} X.$$

Procedure to remove identity blocks: general description

◇ Hyp.: we have computed $Y_{s,j-1} \in \mathrm{GL}_{n-s-j+1}(D)$, $X_{j-1} \in \mathrm{GL}_{m-j+1}(D)$ satisfying $L'_{s,j-1} = Y_{s,j-1}^{-1}\, L_{s,j-1}\, X_{j-1}$.

$$
\begin{array}{ccc}
D^{1\times(n-s-j)} & \xrightarrow{.L'_{s,j}} & D^{1\times(m-j)} \\
\uparrow .G'_{1,j} & & \uparrow .G'_{2,j} \\
D^{1\times(n-s-j+1)} & \xrightarrow{.L'_{s,j-1}} & D^{1\times(m-j+1)} \\
\uparrow .Y_{s,j-1} & & \uparrow .X_{j-1} \\
D^{1\times(n-s-j+1)} & \xrightarrow{.L_{s,j-1}} & D^{1\times(m-j+1)} \\
\uparrow .W_{1,j} & & \uparrow .W_{2,j} \\
D^{1\times(n-s-j+1)} & \xrightarrow{.L_{s,j-1}} & D^{1\times(m-j+1)} \\
\uparrow .H_{1,j} & & \uparrow .H_{2,j} \\
D^{1\times(n-s-j)} & \xrightarrow{.L_{s,j}} & D^{1\times(m-j)}
\end{array}
$$

$$
\Longrightarrow L_{s,j}\, \underbrace{(H_{2,j}\, W_{2,j}\, X_{j-1}\, G'_{2,j})}_{X_j \in \mathrm{GL}_{m-j}(D)} = \underbrace{(H_{1,j}\, W_{1,j}\, Y_{s,j-1}\, G'_{1,j})}_{Y_{s,j} \in \mathrm{GL}_{n-s-j}(D)}\, L'_{s,j}.
$$

Procedure to remove identity blocks: matrix computations

◇ The process is the same as for removing zero rows:

1. We decompose $Y_{s,j-1}$, X_{j-1}, and their inverses by blocks;
2. The key lemma and the assumption $\mathrm{sr}(D) \leq p' - r$ implies that there exist $c \in D$, $u \in D^{p'-j}$, and $v \in D^{1\times(p'-j)}$ s.t., if $k_2 = p - j + 1$,

$$\begin{pmatrix} c\,(X'_{11})_{k_2 .} & v \end{pmatrix} \begin{pmatrix} (X_{11})_{.k_2} \\ (X_{21})_{.k_2} + u\,(X_{31})_{k_2} \end{pmatrix} = 1;$$

3. Explicit formulas for all the matrices are then obtained in terms of a solution $c \in D$, $u \in D^{p'-j}$, and $v \in D^{1\times(p'-j)}$.

Example in the theory of 2D linear elasticity

◇ We have $q = 3$, $p = 2$, $q' = 6$, and $p' = 3$ so that $n = 14$ and $m = 5$.

◇ We have $\mathrm{sr}(\mathbb{Q}[\partial_1, \partial_2]) = 3$.

◇ The positive integer r has to satisfy:

$$r \leq \min(p, p') = 2, \quad \mathrm{sr}(D) \leq \max(p - r, p' - r) = 3 - r,$$

$\Longrightarrow$ No positive integer r satisfies the hypothesis of Warfield's theorem.

Example in the theory of 2D linear elasticity

$\diamond$ We have $q = 3$, $p = 2$, $q' = 6$, and $p' = 3$ so that $n = 14$ and $m = 5$.

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$\Longrightarrow$ No positive integer r satisfies the hypothesis of Warfield's theorem.

$\diamond$ But, in the above process, the condition on r is just a sufficient condition for $\begin{pmatrix} c\,(X'_{11})_{k_2.} & v \end{pmatrix} \begin{pmatrix} (X_{11})_{.k_2} \\ (X_{21})_{.k_2} + u\,(X_{31})_{k_2} \end{pmatrix} = 1$ to admit a solution!

$\Rightarrow$ In some cases, such c, u, and v could exist without the hypothesis on r.

Example in the theory of 2D linear elasticity

◇ Here, to remove the first identity block, we are reduced to solving:

$$\begin{pmatrix} c & \begin{pmatrix} 0 & 0 \end{pmatrix} & v \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} - u \end{pmatrix} = 1.$$

◇ Even if the hypothesis of Warfield's theorem is not fulfilled, a solution is clearly given by $c = 0$, $v = (0 \quad -1)$, and $u = (0 \quad 1)^T$.

◇ This allows to remove a first identity block!

Example in the theory of 2D linear elasticity

◇ From the explicit formulas, we get $X_1 \in \mathrm{GL}_4(D)$ and $Y_{5,1} \in \mathrm{GL}_8(D)$ so that we have the equivalence of matrices $L'_{5,1} = Y_{5,1}^{-1}\, L'_{5,1}\, X_1$.

◇ Similarly, we can remove a second identity block and we finally get:

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ \hline \partial_1 & 0 & 0 \\ \partial_2 & -1 & 0 \\ 0 & \partial_1 & 0 \\ 0 & 1 & \partial_1 \\ 0 & 0 & \partial_2 \\ 0 & \partial_2 & 0 \end{pmatrix}}_{L'_{5,2}} = Y_{5,2}^{-1} \underbrace{\left(\begin{array}{cc|c} \partial_1 & 0 & 0 \\ \frac{1}{2}\,\partial_2 & \frac{1}{2}\,\partial_1 & 0 \\ 0 & \partial_2 & 0 \\ \hline 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)}_{L_{5,2}} X_2.$$

Summary

◇ We have given constructive versions of Fitting and Warfield's theorems.

◇ Explicit formulas for all unimodular matrices providing the equivalences are given in terms of the matrices defining the D-module isomorphism.

◇ Concerning Warfield's theorem, the method relies on the resolution of a "stable rank" equation.

◇ We have an implementation of all the algorithms in `Maple`. It uses heuristics for solving the "stable rank" equations which allow to treat many examples.

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Thank you for your attention!