The six vertex model on random lattices using Jacobi theta functions

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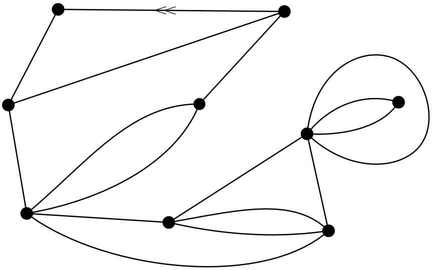
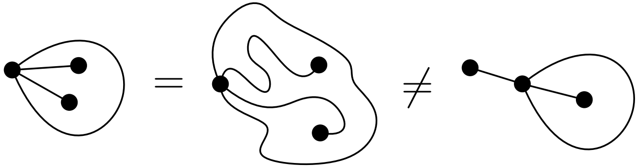
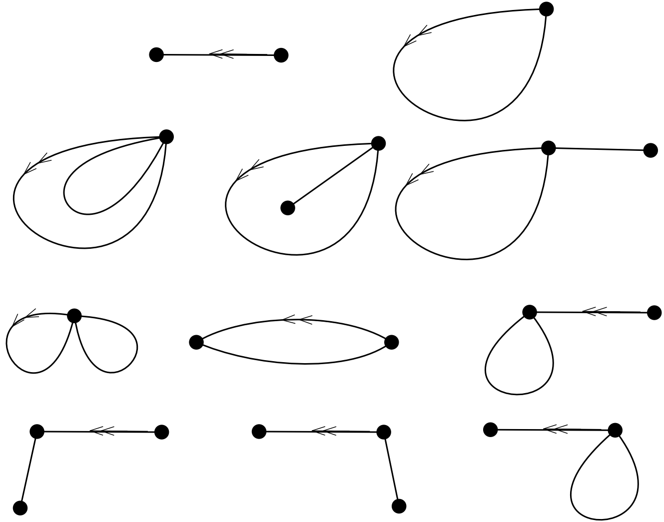
ROOTED PLANAR MAPS

Planar map: Drawing of (planar) graph on the sphere with a marked, directed *root* edge (up to orientation preserving homeomorphisms).

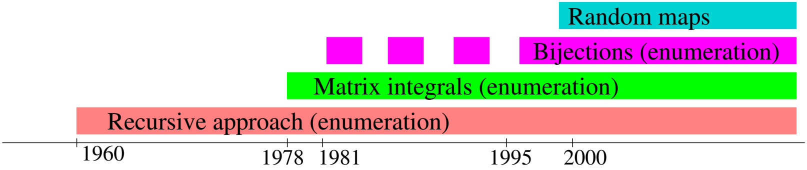




SMALL PLANAR MAPS



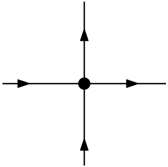
A CHRONOLOGY OF PLANAR MAPS



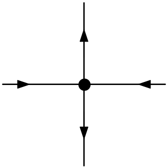
- **Recursive approach:** Tutte, Brown, Bender, Canfield, Richmond, Goulden, Jackson, Wormald, Walsh, Lehman, Gao, Wanless...
- **Matrix integrals:** Brézin, Itzykson, Parisi, Zuber, Bessis, Ginsparg, Kostov, Zinn-Justin, Boulatov, Kazakov, Mehta, Bouttier, Di Francesco, Guitter, Eynard...
- **Bijections:** Cori & Vauquelin, Schaeffer, Bouttier, Di Francesco & Guitter (BDG), Bernardi, Fusy, Poulalhon, Bousquet-Mélou, Chapuy...
- **Geometric properties of random maps:** Chassaing & Schaeffer, BDG, Marckert & Mokkadem, Jean-François Le Gall, Miermont, Curien, Albenque, Bettinelli, Ménard, Angel, Sheffield, Miller, Gwynne...

"SIX" VERTEX MODEL

Each vertex has 2 incoming and 2 outgoing edges.



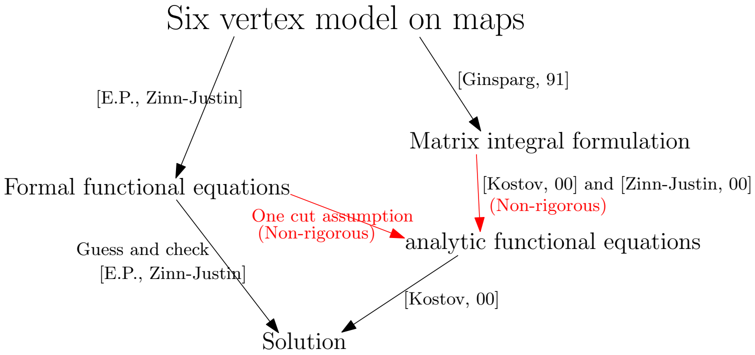
Non-alternating (weight t)



Alternating (weight $t\gamma$)

Definition: $q_{n,k}$ = number of maps with n vertices, k alternating and $n-k$ non-alternating.
Definition: *weight* of a map = product of weights of its vertices.
Definition: $\mathsf{Q}(t,\gamma) = \sum_{n,k} q_{n,k} t^n \gamma^k$ = sum of weights of all maps.
Aim: Determine $\mathsf{Q}(t,\gamma)$.

- "Solved" by Kostov in 2000 using matrix integral techniques.
- Solution was not rigorous.
- We made this argument rigorous and simplified the form of the solutions.



Solutions discovered for $\gamma = 0, 1$ using a completely different method [E.P. and Bousquet-Mélou, 2019]:

The generating function $\mathsf{Q}(t, 0)$ is given by

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n}^2 \mathsf{R}_0(t)^{n+1},$$

$$\mathsf{Q}(t,0) = \frac{1}{2t^2}(t - 2t^2 - \mathsf{R}_0(t)).$$

The generating function $\mathsf{Q}(t, 1)$ is given by

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} \binom{3n}{n} \mathsf{R}_1(t)^{n+1},$$

$$\mathsf{Q}(t,1) = \frac{1}{3t^2}(t - 3t^2 - \mathsf{R}_1(t)).$$

Define

$$\vartheta(z,q)=\sum_{n=0}^{\infty}(-1)^n(e^{(2n+1)iz}-e^{-(2n+1)iz})q^{(2n+1)^2/8}.$$

Let $q=q(t,\alpha)$ be the unique series satisfying

$$t=\frac{\cos\alpha}{64\sin^3\alpha}\left(-\frac{\vartheta(\alpha,q)\vartheta'''(\alpha,q)}{\vartheta'(\alpha,q)^2}+\frac{\vartheta''(\alpha,q)}{\vartheta'(\alpha,q)}\right).$$

Define $\mathsf{R}(t,\gamma)$ by

$$\mathsf{R}(t,-2\cos(2\alpha))=\frac{\cos^2\alpha}{96\sin^4\alpha}\frac{\vartheta(\alpha,q)^2}{\vartheta'(\alpha,q)^2}\left(-\frac{\vartheta'''(\alpha,q)}{\vartheta'(\alpha,q)}+\frac{\vartheta'''(0,q)}{\vartheta'(0,q)}\right).$$

Then

$$\mathsf{Q}(t,\gamma)=\frac{1}{(\gamma+2)t^2}\left(t-(\gamma+2)t^2-\mathsf{R}(t,\gamma)\right).$$

- **Part 1:** Functional equations for six vertex model
- **Part 2:** Solving functional equations (guess and check)
- **Part 3:** deriving the guesses
- **Part 4:** Modular properties and algebraicity for the 6-vertex model

Part 1:
Deriving functional equations

Theorem:
There are unique series

$$\mathsf{H}(x,y) \equiv \mathsf{H}(t,\omega,x,y) \in \mathbb{C}[x,y,\omega,\omega^{-1}][[t]], \qquad \text{and}$$
$$\mathsf{W}(x) \equiv \mathsf{W}(t,\omega,x) \in \mathbb{C}[x,\omega,\omega^{-1}][[t]],$$

satisfying

$$\mathsf{W}(x) = x^2 t\mathsf{W}(x)^2 + \omega x t\mathsf{H}(0,x) + \omega^{-1} x t\mathsf{H}(x,0) + 1,$$
$$\mathsf{H}(x,y) = \mathsf{W}(x)\mathsf{W}(y) + \frac{\omega}{y}\left(\mathsf{H}(x,y) - \mathsf{H}(x,0)\right) + \frac{\omega^{-1}}{x}\left(\mathsf{H}(x,y) - \mathsf{H}(0,y)\right).$$

The series $\mathsf{C}(t,\omega) = \mathsf{Q}(t,\omega^2+\omega^{-2})$ is given by

$$\mathsf{C}(t,\omega) = \mathsf{H}(t,\omega,0,0).$$

Recall: $\mathsf{Q}(t, \gamma)$ counts six-vertex model configurations with a weight t per vertex and γ per alternating vertex.

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Definition: CEPO: a map using vertices of the types below

Vertex types:

Weight t per undirected edge

CUBIC EULERIAN PARTIAL ORIENTATIONS (CEPOS)

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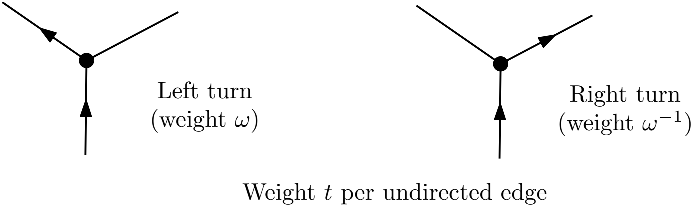

$\mathsf{C}(t,\omega) = \sum_{n,k} c_{n,k} t^n \omega^k$ counts CEPOs (using weights above).
$c_{n,2k}$: number of CEPOs with $n+k$ left turns and $n-k$ right turns

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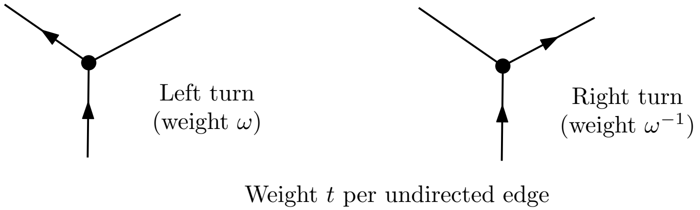

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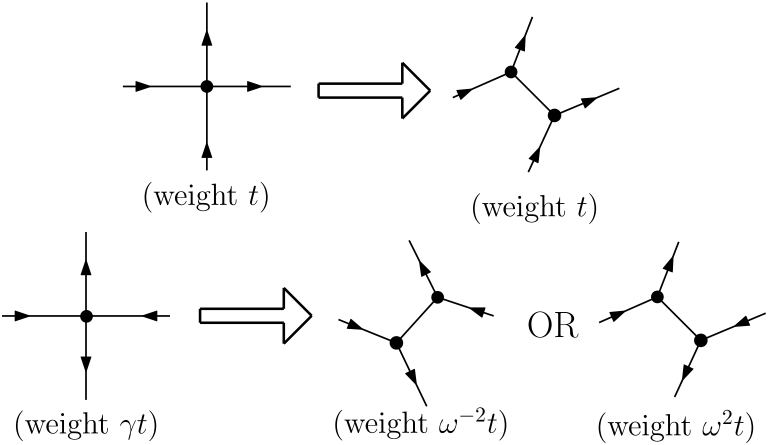
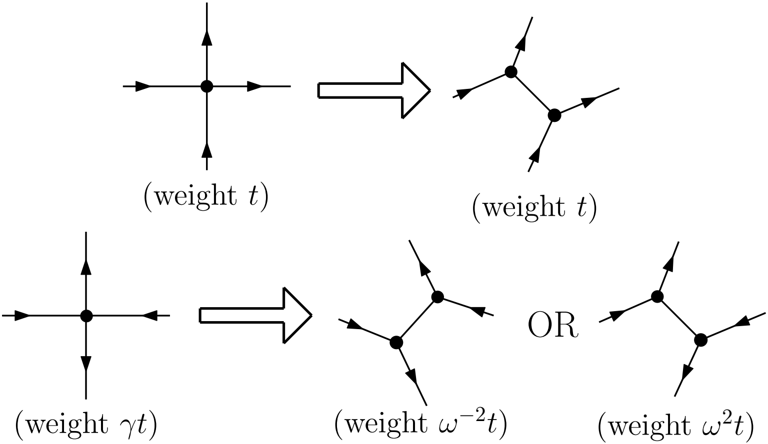


SIX VERTEX MODEL $\to$ CEPOS

Theorem: $\mathsf{Q}(t, \omega^2 + \omega^{-2}) = \mathsf{C}(t, \omega)$



Reverse direction: contract undirected edges.

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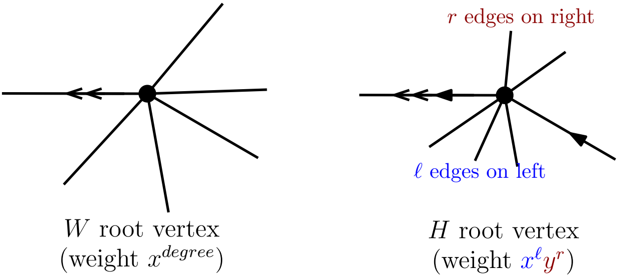

$\mathsf{W}(x) \equiv \mathsf{W}(t, \omega, x)$: root vertex is a W root vertex

$\mathsf{H}(x, y) \equiv \mathsf{H}(t, \omega, x, y)$: root vertex is a H root vertex

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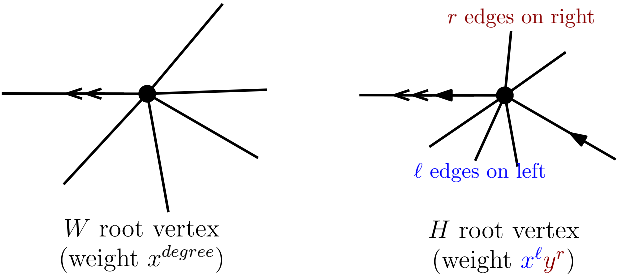

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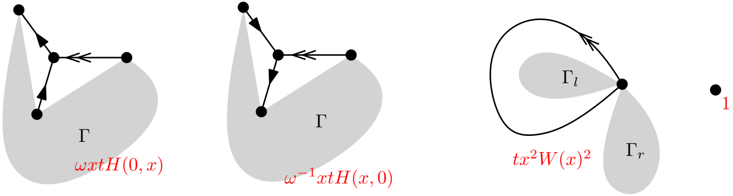
$\mathsf{H}(x,y) \equiv \mathsf{H}(t,\omega,x,y)$: root vertex is a H root vertex

$$\mathsf{C}(t,\omega) = \mathsf{H}(t,\omega,0,0)$$

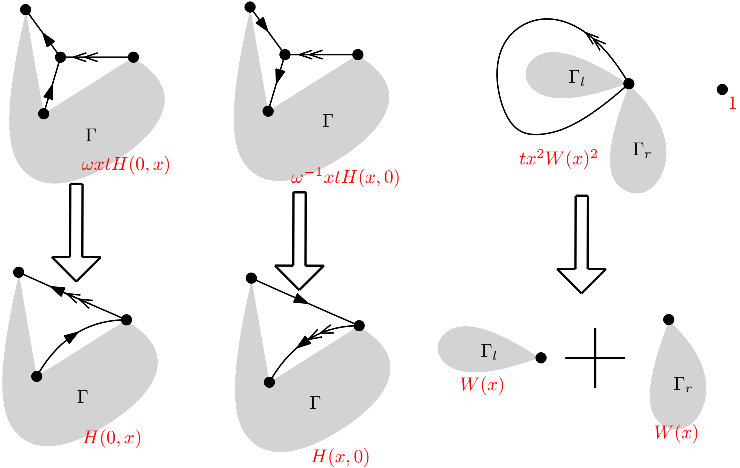
For functional equations: Contract the root edge.

CUBIC EULERIAN PARTIAL ORIENTATIONS



$$\mathsf{W}(x) = \omega xt\mathsf{H}(0,x) + \omega^{-1}xt\mathsf{H}(x,0) + x^2t\mathsf{W}(x)^2 + 1$$

Cubic Eulerian partial orientations



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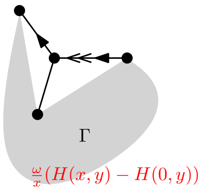

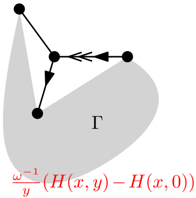

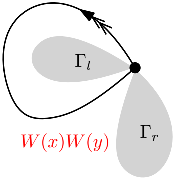

$$\mathsf{H}(x,y) = \frac{\omega}{x}\left(\mathsf{H}(x,y)-\mathsf{H}(0,y)\right)+\frac{\omega^{-1}}{y}\left(\mathsf{H}(x,y)-\mathsf{H}(x,0)\right)+\mathsf{W}(x)\mathsf{W}(y).$$

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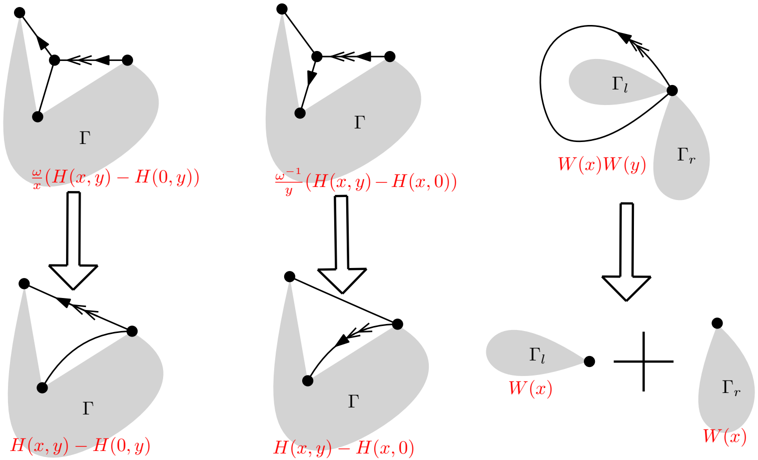

$$\mathsf{H}(x, y) = \frac{\omega}{x}\left(\mathsf{H}(x, y) - \mathsf{H}(0, y)\right) + \frac{\omega^{-1}}{y}\left(\mathsf{H}(x, y) - \mathsf{H}(x, 0)\right) + \mathsf{W}(x)\mathsf{W}(y).$$

Part 2:
Solving the functional equations

Theorem:
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$$\mathsf{W}(x) \equiv \mathsf{W}(t,\omega,x) \in \mathbb{C}[x,\omega,\omega^{-1}][[t]],$$

satisfying

$$\mathsf{W}(x) = x^2 t \mathsf{W}(x)^2 + \omega x t \mathsf{H}(0,x) + \omega^{-1} x t \mathsf{H}(x,0) + 1,$$
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The series $\mathsf{C}(t,\omega) = \mathsf{Q}(t,\omega^2+\omega^{-2})$ is given by

$$\mathsf{C}(t,\omega) = \mathsf{H}(t,\omega,0,0).$$

So, we just need to guess and check

Expressions for $\mathsf{W}(x)$ and $\mathsf{H}(x, y)$

$$\vartheta(z) = \sum_{n=0}^{\infty}(-1)^n(e^{(2n+1)iz} + e^{-(2n+1)iz})e^{(n+1/2)^2 i\pi\tau}$$

$$\omega = ie^{-i\alpha}, \qquad t = \frac{\cos\alpha}{64\sin^3\alpha}\left(-\frac{\vartheta(\alpha)\vartheta'''(\alpha)}{\vartheta'(\alpha)^2} + \frac{\vartheta''(\alpha)}{\vartheta'(\alpha)}\right),$$

$$b = \frac{1}{16t}\frac{\cos(\alpha)}{\sin^3(\alpha)}\frac{\vartheta(\alpha)^2}{\vartheta'(\alpha)^2}, \qquad x_0 = \frac{\cos\alpha}{2\sin\alpha}\frac{\vartheta'(0)}{\vartheta'(\alpha)}, \qquad c = -\omega - \omega^{-1},$$

$$V(z) = b\left(\frac{\vartheta'(z)^2}{\vartheta(z)^2} - \frac{\vartheta''(z)}{\vartheta(z)} + \frac{\vartheta'''(0)}{3\vartheta'(0)}\right), \qquad x(z) = x_0\frac{\vartheta(z+\alpha)}{\vartheta(z)},$$

$$W^{(0)}(y) = -\frac{1}{2\pi}\int_0^{\pi}\frac{V(z-\frac{\pi\tau}{2})x'\left(z-\frac{\pi\tau}{2}\right)}{\left(y+c^{-1}-i\omega x(z-\frac{\pi\tau}{2})\right)x\left(z-\frac{\pi\tau}{2}\right)}dz,$$

$$\frac{1}{y}\mathsf{H}\left(0, \frac{1}{y}\right) = -\frac{\omega^{-1}}{2\pi}\int_0^{\pi}\frac{V(z-\frac{\pi\tau}{2})x'\left(z-\frac{\pi\tau}{2}\right)W^{(0)}\left(-c^{-1}-i\omega^{-1}x\left(z-\frac{\pi\tau}{2}\right)\right)}{(y+c^{-1}-i\omega x(z-\frac{\pi\tau}{2}))x\left(z-\frac{\pi\tau}{2}\right)}dz,$$

$$\frac{1}{y}\mathsf{H}\left(\frac{1}{y}, 0\right) = -\frac{\omega}{2\pi}\int_0^{\pi}\frac{V(z-\frac{\pi\tau}{2})x'\left(z-\frac{\pi\tau}{2}\right)W^{(0)}\left(-c^{-1}-i\omega^{3}x\left(z-\frac{\pi\tau}{2}\right)\right)}{(y+c^{-1}-i\omega x(z-\frac{\pi\tau}{2}))x\left(z-\frac{\pi\tau}{2}\right)}dz,$$

$$\mathsf{W}(x) = \frac{1}{x}W^{(0)}\left(\frac{1}{x}\right), \qquad \mathsf{H}(x, y) = \frac{xy\mathsf{W}(x)\mathsf{W}(y) - \omega x\mathsf{H}(x, 0) - \omega^{-1}y\mathsf{H}(0, y)}{xy - \omega x - \omega^{-1}y}.$$

Expressions are formal series in t, ω, and (in some cases) x, y and e^{2iz}.

Solution for $\mathsf{Q}(t,\gamma)$

Define

$$\vartheta(z,q) = \sum_{n=0}^{\infty} (-1)^n \left(e^{(2n+1)iz} - e^{-(2n+1)iz}\right) q^{(2n+1)^2/8}.$$

Let $q = q(t,\alpha)$ be the unique series satisfying

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Define $\mathsf{R}(t,\gamma)$ by

$$\mathsf{R}(t,-2\cos(2\alpha)) = \frac{\cos^2\alpha}{96\sin^4\alpha}\frac{\vartheta(\alpha,q)^2}{\vartheta'(\alpha,q)^2}\left(-\frac{\vartheta'''(\alpha,q)}{\vartheta'(\alpha,q)} + \frac{\vartheta'''(0,q)}{\vartheta'(0,q)}\right).$$

Then

$$\mathsf{Q}(t,\gamma) = \frac{1}{(\gamma+2)t^2}\left(t - (\gamma+2)t^2 - \mathsf{R}(t,\gamma)\right).$$

Part 3: Non-rigorously deriving expressions for $\mathsf{W}(x)$ and $\mathsf{H}(x, y)$

(by copying Kostov)

"Solving" functional equations

Recall: Equations to solve:

$$\mathsf{W}(x) = x^2 t\mathsf{W}(x)^2 + \omega xt\mathsf{H}(0,x) + \omega^{-1}xt\mathsf{H}(x,0) + 1$$

$$\mathsf{H}(x,y) = \mathsf{W}(x)\mathsf{W}(y) + \frac{\omega}{y}\left(\mathsf{H}(x,y) - \mathsf{H}(x,0)\right) + \frac{\omega^{-1}}{x}\left(\mathsf{H}(x,y) - \mathsf{H}(0,y)\right).$$

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Step 1: Write $\omega = e^{i\alpha}$ for $\alpha \in \mathbb{R}$ and choose $t \in \mathbb{R}$ small
$\rightarrow$ series converge for $|x|, |y| < 1$

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Step 2: (One cut assumption) Assume $\mathsf{W}(x)$ and $\mathsf{H}(x,0)$ have extensions that are analytic on $\mathbb{C} \setminus [r_1, r_2]$, for some $r_1, r_2 \in \mathbb{R}$

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Step 3: (Kernel method) Substitute $y \rightarrow \frac{\omega^2 x}{\omega x - 1}$ so that $\left(1 - \frac{\omega}{y} - \frac{\omega^{-1}}{x}\right) = 0$. The second equation becomes:

$$0 = \mathsf{W}(x)\mathsf{W}\left(\frac{\omega^2 x}{\omega x - 1}\right) - \frac{\omega x - 1}{\omega x}\mathsf{H}(x,0) - \frac{\omega^{-1}}{x}\mathsf{H}\left(0, \frac{\omega^2 x}{\omega x - 1}\right)$$

“SOLVING” FUNCTIONAL EQUATIONS

New equations: (with $\mathsf{W}(x)$, $\mathsf{H}(x,0)$, $\mathsf{H}(0,x)$ analytic on $\mathbb{C}\setminus[r_1,r_2]$)

$$\mathsf{W}(x) = x^2 t\mathsf{W}(x)^2 + \omega x t\mathsf{H}(0,x) + \omega^{-1} x t\mathsf{H}(x,0) + 1$$

$$0 = \mathsf{W}(x)\mathsf{W}\left(\frac{\omega^2 x}{\omega x - 1}\right) - \frac{\omega x - 1}{\omega x}\mathsf{H}(x,0) - \frac{\omega^{-1}}{x}\mathsf{H}\left(0, \frac{\omega^2 x}{\omega x - 1}\right)$$

$$0 = \mathsf{W}\left(\frac{\omega^{-1} x}{x - \omega}\right)\mathsf{W}(x) - \frac{\omega}{x}\mathsf{H}\left(\frac{\omega^{-1} x}{x - \omega}, 0\right) - \frac{x - \omega}{x}\mathsf{H}(0,x)$$

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$$0 = \mathsf{W}(x)\mathsf{W}\left(\frac{\omega^2x}{\omega x-1}\right) - \frac{\omega x-1}{\omega x}\mathsf{H}(x,0) - \frac{\omega^{-1}}{x}\mathsf{H}\left(0,\frac{\omega^2x}{\omega x-1}\right)$$

$$0 = \mathsf{W}\left(\frac{\omega^{-1}x}{x-\omega}\right)\mathsf{W}(x) - \frac{\omega}{x}\mathsf{H}\left(\frac{\omega^{-1}x}{x-\omega},0\right) - \frac{x-\omega}{x}\mathsf{H}(0,x)$$

For $x = r \in [r_1,r_2]$, taking the difference between the expressions above and below the cut yields:

$$0 = -(\mathsf{W}^+(r)-\mathsf{W}^-(r)) + r^2t(\mathsf{W}^+(r)^2-\mathsf{W}^-(r)^2) + \omega rt(\mathsf{H}^+(0,r)-\mathsf{H}^-(0,r)) + \omega^{-1}rt(\mathsf{H}^+(r,0)-\mathsf{H}^-(r,0))$$

$$0 = (\mathsf{W}^+(r)-\mathsf{W}^-(r))\mathsf{W}\left(\frac{\omega^2r}{\omega r-1}\right) - \frac{\omega r-1}{\omega r}(\mathsf{H}^+(r,0)-\mathsf{H}^-(r,0))$$

$$0 = \mathsf{W}\left(\frac{\omega^{-1}r}{r-\omega}\right)(\mathsf{W}^+(r)-\mathsf{W}^-(r)) - \frac{r-\omega}{r}(\mathsf{H}^+(0,r)-\mathsf{H}^-(0,r))$$

New equations:

$$0 = -(\mathsf{W}^+(r) - \mathsf{W}^-(r)) + r^2t(\mathsf{W}^+(r)^2 - \mathsf{W}^-(r)^2) + \omega rt(\mathsf{H}^+(0,r) - \mathsf{H}^-(0,r)) + \omega^{-1}rt(\mathsf{H}^+(r,0) - \mathsf{H}^-(r,0))$$

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Combining these yields:

$$0 = -(\mathsf{W}^+(r) - \mathsf{W}^-(r)) + r^2t(\mathsf{W}^+(r)^2 - \mathsf{W}^-(r)^2) + \frac{\omega r^2 t}{r-\omega}\mathsf{W}\left(\frac{\omega^{-1} r}{r - \omega}\right)(\mathsf{W}^+(r) - \mathsf{W}^-(r)) + \frac{r^2 t}{\omega r - 1}(\mathsf{W}^+(r) - \mathsf{W}^-(r))\mathsf{W}\left(\frac{\omega^2 r}{\omega r - 1}\right)$$

New equations:

$$0 = -(\mathsf{W}^+(r) - \mathsf{W}^-(r)) + r^2t(\mathsf{W}^+(r)^2 - \mathsf{W}^-(r)^2)$$
$$+\,\omega rt(\mathsf{H}^+(0,r) - \mathsf{H}^-(0,r)) + \omega^{-1}rt(\mathsf{H}^+(r,0) - \mathsf{H}^-(r,0))$$

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Combining these yields: Now divide by $(\mathsf{W}^+(r) - \mathsf{W}^-(r))$

$$0 = -(\mathsf{W}^+(r) - \mathsf{W}^-(r)) + r^2t(\mathsf{W}^+(r)^2 - \mathsf{W}^-(r)^2)$$
$$+\,\frac{\omega r^2 t}{r - \omega}\mathsf{W}\left(\frac{\omega^{-1} r}{r - \omega}\right)(\mathsf{W}^+(r) - \mathsf{W}^-(r))$$
$$+\,\frac{r^2 t}{\omega r - 1}(\mathsf{W}^+(r) - \mathsf{W}^-(r))\mathsf{W}\left(\frac{\omega^2 r}{\omega r - 1}\right)$$

New equations:

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Combining these yields: Now divide by $(\mathsf{W}^+(r) - \mathsf{W}^-(r))$

$$0 = -1 + r^2t(\mathsf{W}^+(r) + \mathsf{W}^-(r)) + \frac{\omega r^2 t}{r - \omega}\mathsf{W}\left(\frac{\omega^{-1} r}{r - \omega}\right) + \frac{r^2 t}{\omega r - 1}\mathsf{W}\left(\frac{\omega^2 r}{\omega r - 1}\right)$$

New equations:

$$0 = -(\mathsf{W}^+(r) - \mathsf{W}^-(r)) + r^2t(\mathsf{W}^+(r)^2 - \mathsf{W}^-(r)^2) + \omega rt(\mathsf{H}^+(0,r) - \mathsf{H}^-(0,r)) + \omega^{-1}rt(\mathsf{H}^+(r,0) - \mathsf{H}^-(r,0))$$

$$0 = (\mathsf{W}^+(r) - \mathsf{W}^-(r))\mathsf{W}\left(\frac{\omega^2 r}{\omega r - 1}\right) - \frac{\omega r - 1}{\omega r}(\mathsf{H}^+(r,0) - \mathsf{H}^-(r,0))$$

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Now, writing $X(v) = \frac{\omega+\omega^{-1}}{1-iv(\omega^2+1)}$ and $Y(v) = \frac{\omega+\omega^{-1}}{1-iv(\omega^{-2}+1)}$, the function

$$U(v) := v\omega X(v)\mathsf{W}(X(v)) + v\omega^{-1}Y(v)\mathsf{W}(Y(v)) + \frac{iv^2}{t(\omega^2-\omega^{-2})} - \frac{v}{t(\omega+\omega^{-1})^2}$$

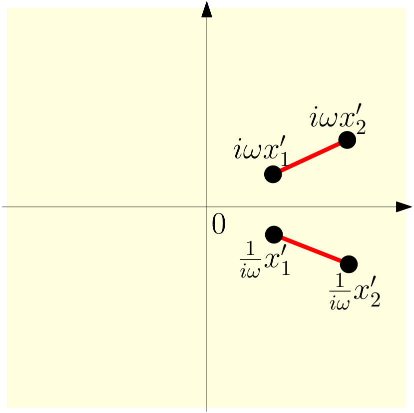
is analytic except on 2 cuts $i\omega[x_1', x_2']$ and $-i\omega^{-1}[x_1', x_2']$ and satisfies

$$U(i\omega(x+i0)) = U(-i\omega^{-1}(x-i0)),$$

$$U(i\omega(x-i0)) = U(-i\omega^{-1}(x+i0)),$$

for $x \in [x_1', x_2']$.

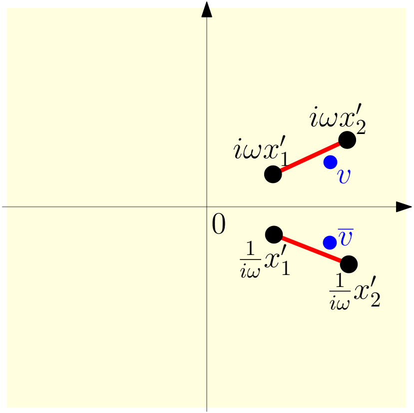
UNDERSTANDING $U(v)$



$U(v)$ analytic except on slits .

$U(i\omega(x \pm i0)) = U(-i\omega^{-1}(x \mp i0))$, for $x \in [x'_1, x'_2]$.
i.e., as $v \to$ slit, $U(v) - U(\overline{v}) \to 0$.

UNDERSTANDING $U(v)$



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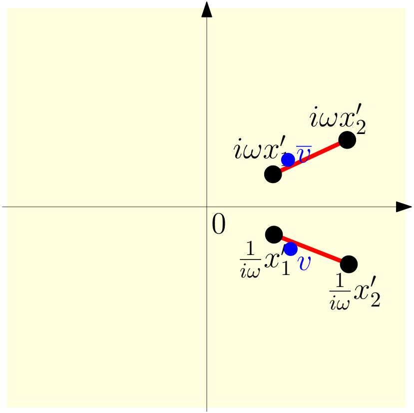

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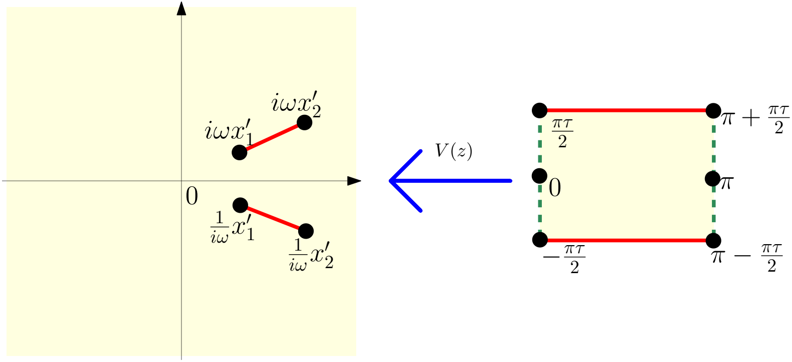
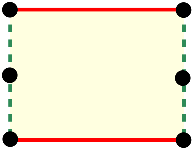
SOLVING FOR $U(v)$

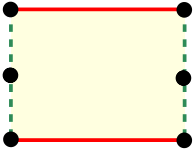


Left: $U(v)$ is analytic on yellow region.
There is a unique $\tau \in i\mathbb{R}_{>0}$ and conformal map $V(z)$ from the flat cylinder of height $\pi\tau$ onto this region ($V(z) = V(z+\pi)$).

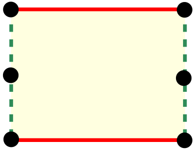
Solving for $U(v)$

Solving for $U(v)$

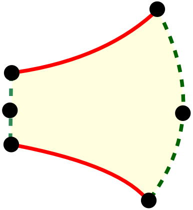




SOLVING FOR $U(v)$



Solving for $U(v)$



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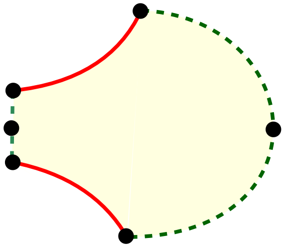

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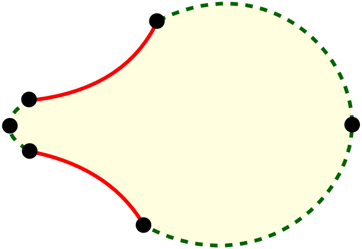

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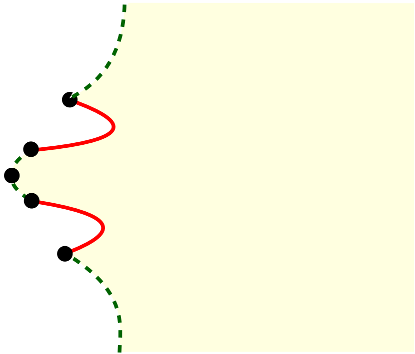
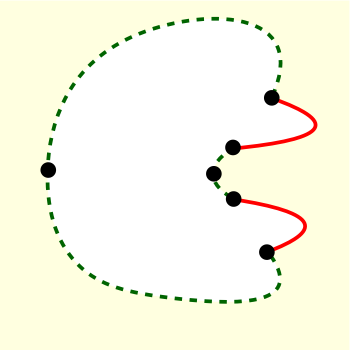
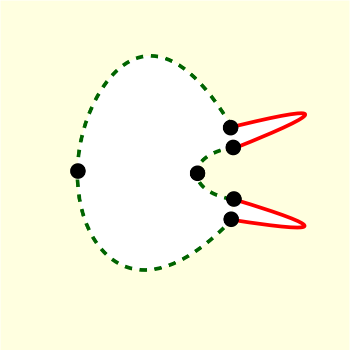


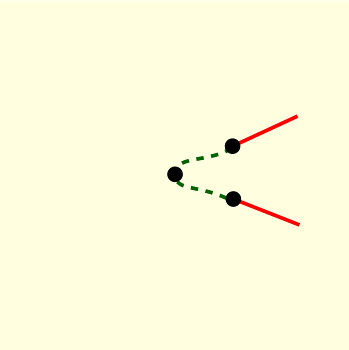
SOLVING FOR $U(v)$



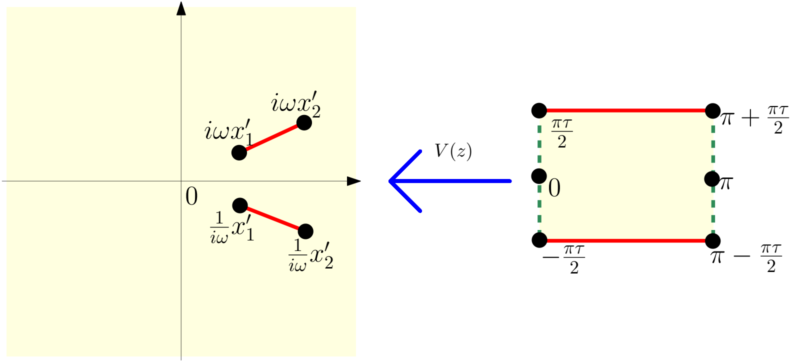
SOLVING FOR $U(v)$



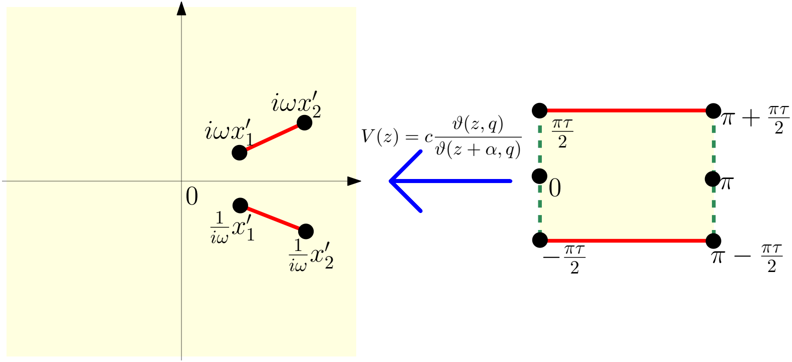
SOLVING FOR $U(v)$



SOLVING FOR $U(v)$



Left: $U(v)$ is analytic on yellow region.

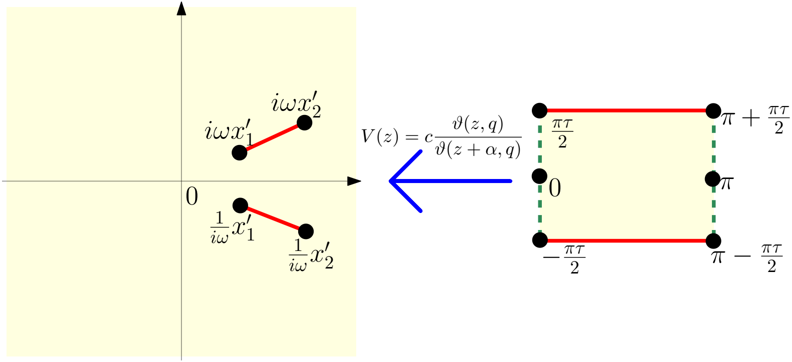


Left: $U(v)$ is analytic on yellow region.

$$V(z) = c\frac{\vartheta(z,q)}{\vartheta(z+\alpha,q)},$$

where $\omega = ie^{i\alpha}$ and $q = e^{2\pi i\tau}$.

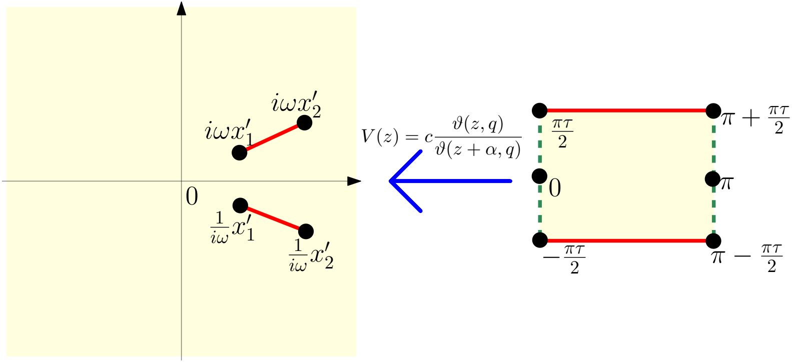
Solving for $U(v)$



$$U(i\omega(x \pm i0)) = U(-i\omega^{-1}(x \mp i0))$$

$$\Rightarrow \qquad U(V(z + \frac{\pi\tau}{2})) = U(V(z - \frac{\pi\tau}{2}))$$

Solving for $U(v)$



$$U(i\omega(x \pm i0)) = U(-i\omega^{-1}(x \mp i0))$$

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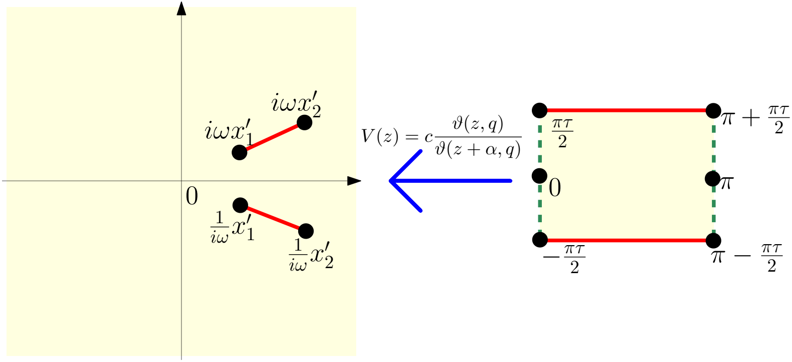
SOLVING FOR $U(v)$

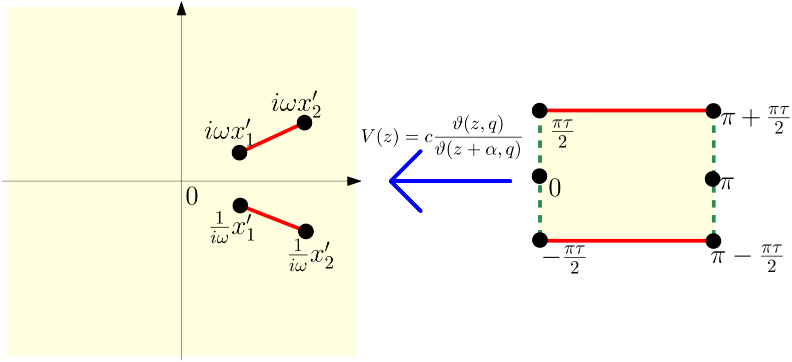


$$U(i\omega(x \pm i0)) = U(-i\omega^{-1}(x \mp i0))$$

$$\Rightarrow \quad U(V(z+\pi\tau)) = U(V(z))$$

$$\Rightarrow \quad U(V(z)) = A + B\wp(z+\alpha,\tau)$$

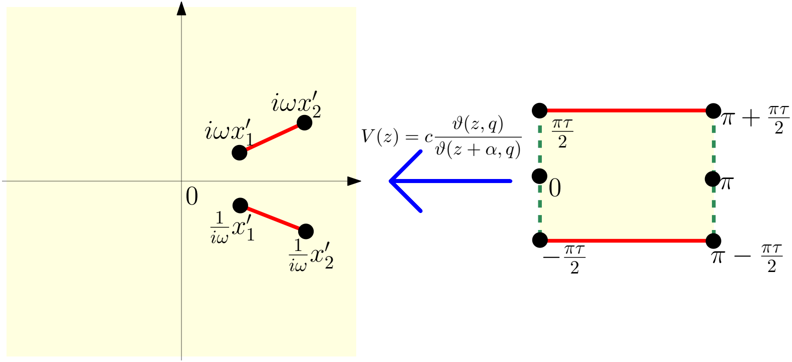
Solving for $U(v)$



$$
\begin{aligned}
& U(i\omega(x \pm i0)) = U(-i\omega^{-1}(x \mp i0)) \\
\Rightarrow \quad & U(V(z+\pi\tau)) = U(V(z)) \\
\Rightarrow \quad & U(V(z)) = A + B\wp(z+\alpha,\tau)
\end{aligned}
$$

Hooray, it's solved!

SOLVING FOR $U(v)$



$$U(i\omega(x \pm i0)) = U(-i\omega^{-1}(x \mp i0))$$
$$\Rightarrow \quad U(V(z+\pi\tau)) = U(V(z))$$
$$\Rightarrow \quad U(V(z)) = A + B\wp(z+\alpha,\tau)$$

$\rightarrow$ integral expression for $\mathsf{W}(x)$ and $\mathsf{H}(x,y) \rightarrow \mathsf{Q}(t,\gamma)$.

Solution for $\mathsf{Q}(t,\gamma)$

Define

$$\vartheta(z,q) = \sum_{n=0}^{\infty} (-1)^n \left(e^{(2n+1)iz} - e^{-(2n+1)iz}\right) q^{(2n+1)^2/8}.$$

Let $q = q(t,\alpha)$ be the unique series satisfying

$$t = \frac{\cos\alpha}{64\sin^3\alpha}\left(-\frac{\vartheta(\alpha,q)\vartheta'''(\alpha,q)}{\vartheta'(\alpha,q)^2} + \frac{\vartheta''(\alpha,q)}{\vartheta'(\alpha,q)}\right).$$

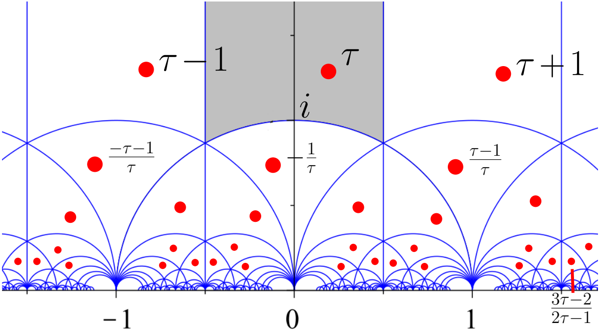
Define $\mathsf{R}(t,\gamma)$ by

$$\mathsf{R}(t,-2\cos(2\alpha)) = \frac{\cos^2\alpha}{96\sin^4\alpha}\frac{\vartheta(\alpha,q)^2}{\vartheta'(\alpha,q)^2}\left(-\frac{\vartheta'''(\alpha,q)}{\vartheta'(\alpha,q)} + \frac{\vartheta'''(0,q)}{\vartheta'(0,q)}\right).$$

Then

$$\mathsf{Q}(t,\gamma) = \frac{1}{(\gamma+2)t^2}\left(t - (\gamma+2)t^2 - \mathsf{R}(t,\gamma)\right).$$

Part 4:
Modular properties in special cases



Nice reference for modular properties of theta functions:

Elliptic Modular Forms and Their Applications, Zagier, 2008.

Recall: $\vartheta(z|\tau) = \vartheta(z, e^{i\pi\tau}) = \sum_{n=0}^{\infty}(-1)^n(e^{(2n+1)iz} + e^{-(2n+1)iz})e^{(n+1/2)^2 i\pi\tau}$

Aim: relate $\vartheta(z|\tau)$ to other τ values

Natural transformations: $\tau \to \tau + 1$ and $\tau \to -\frac{1}{\tau}$

Equations:

- $\vartheta(z|\tau + 1) = e^{i\pi/4}\vartheta(z, \tau)$
- $\vartheta\left(\frac{z}{\tau}| - \frac{1}{\tau}\right) = -i(-i\tau)^{\frac{1}{2}} \exp\left(\frac{i}{\pi\tau}z^2\right)\vartheta(z|\tau)$

These transformations generate the group of transformations

$$\tau \to \frac{a\tau + b}{c\tau + d},$$

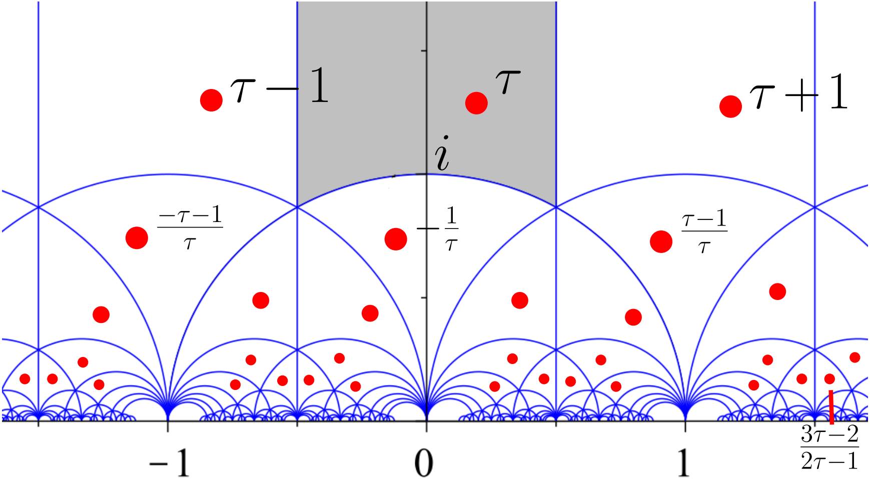
satisfying $ad - bc = 1$.

This is isomorphic to the group $SL_2(\mathbb{Z})$ of matrices

$$\begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix},$$

with determinant 1.

Orbit of $SL_2(\mathbb{Z})$



Definition: $SL_2(\mathbb{Z})$ is the group of matrices $\begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix}$ with determinant 1.

Action on upper half plane $\mathbb{H} = \{z \in \mathbb{C} | \mathbf{im}(z) > 0\}$**:**

$$\begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

Modular functions

Definition: $SL_2(\mathbb{Z})$ is the group of matrices $\begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix}$ with determinant 1.

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Definition: Let Γ be a finite index subgroup of $SL_2(\mathbb{Z})$. A *modular function* is a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following for all $\rho \in \Gamma$:

$$f(\rho \cdot \tau) := f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau).$$

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$$f(\rho \cdot \tau) := f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau).$$

Theorem: (classical) All modular functions are algebraically related

ALGEBRAICITY FOR $\mathsf{Q}(t,\gamma)$??

Recall:

$$t = \frac{\cos\alpha}{64\sin^3\alpha}\left(-\frac{\vartheta(\alpha,\tau)\vartheta'''(\alpha,\tau)}{\vartheta'(\alpha,\tau)^2}+\frac{\vartheta''(\alpha,\tau)}{\vartheta'(\alpha,\tau)}\right).$$

$$\mathsf{R}(t,-2\cos(2\alpha)) = \frac{\cos^2\alpha}{96\sin^4\alpha}\frac{\vartheta(\alpha,\tau)^2}{\vartheta'(\alpha,\tau)^2}\left(-\frac{\vartheta'''(\alpha,\tau)}{\vartheta'(\alpha,\tau)}+\frac{\vartheta'''(0,\tau)}{\vartheta'(0,\tau)}\right).$$

$$\mathsf{Q}(t,\gamma) = \frac{1}{(\gamma+2)t^2}\left(t-(\gamma+2)t^2-\mathsf{R}(t,\gamma)\right).$$

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$$\mathsf{Q}(t,\gamma) = \frac{1}{(\gamma+2)t^2}\left(t-(\gamma+2)t^2-\mathsf{R}(t,\gamma)\right).$$

Theorem [E.P. and Zinn-Justin]: if $\alpha \in \pi\mathbb{Q}$, then R and

$$\mathsf{S} = \frac{1}{t}\frac{d^2 t}{d\mathsf{R}^2}$$

are both modular functions (when written as functions of τ), so they are algebraically related.

$$\mathsf{Q}(t,\gamma) = \frac{1}{(\gamma+2)t^2}\left(t - (\gamma+2)t^2 - \mathsf{R}(t,\gamma)\right)$$

Specific cases:

- $\gamma = 0$: $\quad \dfrac{1}{t}\dfrac{d^2t}{d\mathsf{R}^2} = \dfrac{4}{\mathsf{R}(1-16\mathsf{R})}$.
- $\gamma = 1$: $\quad \dfrac{1}{t}\dfrac{d^2t}{d\mathsf{R}^2} = \dfrac{6}{\mathsf{R}(1-27\mathsf{R})}$.
- $\gamma = -1$: $\quad \dfrac{1}{t}\dfrac{d^2t}{d\mathsf{R}^2} = \dfrac{2}{h(1+h)(1+4h)(1-8h)}$, where $\mathsf{R} = h(1+2h)$.
- $\gamma = \frac{1+\sqrt{5}}{2}$: $\quad \mathsf{R}$ and $\mathsf{S} = \dfrac{1}{t}\dfrac{d^2t}{d\mathsf{R}^2}$ are related by

$$\mathsf{R} = h\left(1 - \frac{1+\sqrt{5}}{2}h\right) / \left(1+(2+\sqrt{5})h\right)^3$$

$$\mathsf{S} = (5+\sqrt{5})\left(1+(2+\sqrt{5})h\right)^6 / \left(h\left(1-\frac{11-5\sqrt{5}}{2}h\right)\left(1-\frac{11+5\sqrt{5}}{2}h\right)^2\left(1-\frac{\sqrt{5}-1}{2}h\right)\right)$$

THE CASES $\gamma = 0, 1$

for $\gamma = 0$: the equation $\frac{1}{t}\frac{d^2t}{d\mathsf{R}^2} = \frac{4}{\mathsf{R}(1-16\mathsf{R})}$ implies

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n}^2 \mathsf{R}(t,0)^{n+1},$$

for $\gamma = 1$: the equation $\frac{1}{t}\frac{d^2t}{d\mathsf{R}^2} = \frac{6}{\mathsf{R}(1-27\mathsf{R})}$ implies

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n}\binom{3n}{n} \mathsf{R}(t,1)^{n+1}$$

Recall: in each case

$$\mathsf{Q}(t,\gamma) = \frac{1}{(\gamma+2)t^2}\left(t - (\gamma+2)t^2 - \mathsf{R}(t,\gamma)\right).$$

QUESTIONS

- Can we use the method for $\gamma = 0, 1$ to solve for general gamma?

- Are $\mathsf{W}(x)$ and $\mathsf{H}(x, y)$ D-algebraic?
- Can anyone solve this equation for $a \neq 1$:

$$0 = -1 + r^2 t(\mathsf{W}^+(r) + \mathsf{W}^-(r)) + a\left(\frac{\omega r^2 t}{r-\omega}\mathsf{W}\left(\frac{\omega^{-1} r}{r-\omega}\right) + \frac{r^2 t}{\omega r - 1}\mathsf{W}\left(\frac{\omega^2 r}{\omega r - 1}\right)\right)$$

- Can we use the method for $\gamma = 0, 1$ to solve for general gamma? Yes! [E.P., Bousquet-Mélou, 2021+]
- Are $\mathsf{W}(x)$ and $\mathsf{H}(x, y)$ D-algebraic?
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$$0 = -1 + r^2 t(\mathsf{W}^+(r) + \mathsf{W}^-(r)) + a\left(\frac{\omega r^2 t}{r - \omega}\mathsf{W}\left(\frac{\omega^{-1} r}{r - \omega}\right) + \frac{r^2 t}{\omega r - 1}\mathsf{W}\left(\frac{\omega^2 r}{\omega r - 1}\right)\right)$$

Thank you!