Singularities of meromorphic $\mathfrak{sl}_2(\mathbb{C})$-connections over Riemann surfaces and their deformations

Martin Klimeš

FER, University of Zagreb

Journées GDR EFI, Strasbourg, May 31–June 2, 2021

Supported by FRABDYN, Croatian Science Foundation project PZS 3055

Meromorphic connection over a Riemann surface

Locally near a pole:

$$\nabla(x) = d - \frac{A(x)}{x^{k+1}}dx, \qquad A(0) \neq 0, \quad k = \text{Poincaré rank}.$$

Linear differential system: $x^{k+1}\frac{d}{dx}y = A(x)y$.

Transformations: formal / analytic

- *gauge transformations:*

$$y \mapsto T(x)y, \qquad \det T(0) \neq 0,$$

- *coordinate–gauge transformations:*

$$x \mapsto \phi(x), \quad y \mapsto T(x)y, \qquad \tfrac{d}{dx}\phi(0) \neq 0, \quad \det T(0) \neq 0,$$

Local analytic classification

Birkhoff (1913), Hukuhara (1937), Turittin (1955), Levelt (1961), Sibuya (1967), Malgrange (1971), Balser, Jurkat, Lutz (1979), Ramis (1985), ...

Formal invariants:

- non-resonant case: polar parts of the eigenvalues of $A(x)\frac{dx}{x^{k+1}}$
  meromorphic 1-forms

$$(\lambda_i^{(0)} + \ldots + \lambda_i^{(k)} x^k)\frac{dx}{x^{k+1}},$$

- general case [Balser, Jurkat, Lutz (1979)]: canonical form of formal fundamental solution matrix

$$\hat{F}(x)\, P(x^{-1})\, x^K x^J U\, e^{Q(x^{-1})}.$$

Analytic invariants: collection of *Stokes matrices* modulo conjugation by diagonal matrices from the exponential torus.

Traceless meromorphic connection on rank 2 vector bundle

Locally: $(*)\quad x^{k+1}\frac{d}{dx}y = A(x)y, \qquad A(x)\in\mathfrak{sl}_2(\mathbb{C}),\ A(0)\neq 0.$

Lemma

$(*)$ is analytically gauge equivalent to a companion system ("meromorphic $\mathfrak{sl}_2(\mathbb{C})$-oper")

$$(\tilde{*})\qquad x^{k+1}\frac{d}{dx}y = \begin{pmatrix} 0 & 1\\ Q(x) & 0\end{pmatrix} y,$$

associated to the second order LDE $(**)\quad \left(x^{k+1}\frac{d}{dx}\right)^{\circ 2} y_1 = Q(x)\, y_1.$

Linear point transformations preserving LDE's $(**)$:

$$x\mapsto\phi(x),\quad y_1\mapsto\psi^{\frac12}(x)\,y_1,\quad \phi^*\left(x^{k+1}\frac{d}{dx}\right)=\psi(x)\,x^{k+1}\frac{d}{dx}.$$

Associated transformations of $(\tilde{*})$: *point transformations*

$$x\mapsto\phi(x),\quad y\mapsto\begin{pmatrix}\psi^{\frac12} & 0\\ \frac12 x^{k+1}\frac{d}{dx}\psi & \psi^{-\frac12}\end{pmatrix} y.$$

Theorem (K.)

Two germs of companion systems $(\tilde{*})$ are formally/analytically coordinate–gauge equivalent if and only if they are formally/analytically point equivalent.

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Quadratic differential

Meromorphic quadratic differential

$$Q(x)\left(\tfrac{dx}{x^{k+1}}\right)^2 = -\det\left[\begin{pmatrix} 0 & 1 \\ Q(x) & 0 \end{pmatrix}\tfrac{dx}{x^{k+1}}\right].$$

Square residue

$$\mu = \operatorname{res}^2_{x=0} Q(x)\left(\tfrac{dx}{x^{k+1}}\right)^2 := \left(\operatorname{res}_{x=0}\sqrt{Q(s)}\tfrac{dx}{x^{k+1}}\right)^2.$$

Denote $m = \operatorname{ord}_{x=0} Q(x)$. If m is odd, then $\mu = 0$.

Lemma (see Strebel (1984))

Analytic normal form of $Q(x)\left(\tfrac{dx}{x^{k+1}}\right)^2$ *w.r.t. coordinate transformations* $x \mapsto \phi(x)$*:*

- $m < 2k$: $\left(1 \pm \sqrt{\mu}\, x^{k-\frac{m}{2}}\right)^2 \left(\frac{dx}{x^{k-\frac{m}{2}+1}}\right)^2$,
- $m = 2k$: $\mu\left(\frac{dx}{x}\right)^2$,
- $m > 2k$: $x^{m-2k}(dx)^2$.

Formal classification

Theorem (K.)

Two companion systems $(\tilde{*})\quad x^{k+1}\frac{d}{dx}y = \begin{pmatrix} 0 & 1 \\ Q(x) & 0 \end{pmatrix} y,\quad m = \operatorname{ord}_{x=0} Q(x),$
with the same $k \geq 0$ *are:*

formally gauge equivalent iff

$m < 2k$ *(irregular):*

same $\begin{cases} j^k Q(x), & m = 0, \\ j^{k+m-1} Q(x), & m > 0, \end{cases}$

$m = 2k$ *(regular):*

same $\begin{cases} j^0 Q(x), & m = 0, \\ j^{3k-1} Q(x), & m > 0, \end{cases}$

(and conjugated monodromy)

$2k < m \leq 3k$ *(regular):*

same $j^{3k} Q(x)$,

$m > 3k$ *(regular):*

always.

formally coordinate–gauge equivalent iff

$m < 2k$ *(irregular):*

equivalent $Q(x)\left(\frac{dx}{x^{k+1}}\right)^2$

$\Leftrightarrow$ *same* m *and* $\operatorname{res}^2 Q(x)\left(\frac{dx}{x^{k+1}}\right)^2$,

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$\Leftrightarrow$ *same* m,

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always.

Universal isomonodromic deformation (Heu (2010))

A family of traceless meromorphic connections $\big(E_t \to X,\ \nabla_t(x)\big)_{t\in(T,0)}$ on rank 2 vector bundles over a Riemann surface X is *isomonodromic* if they are restrictions of some *flat* meromorphic connection $\big(E \to X \times (T,0),\ \nabla(t,x)\big)$.

Transversality condition (TC): locally $(*)$ $\quad \nabla(x,t) = d - \frac{A(x,t)}{x^{k+\mathbf{1}}}dx - \frac{B(x,t)}{x^k}dt.$

Lemma (Heu (2010))

In the $\mathfrak{sl}_2(\mathbb{C})$ case, assuming (TC) *and that the Katz rank $k - \frac{m}{2}$ is locally constant along the polar divisor, then $(*)$ is locally analytically coordinate–gauge equivalent to the constant deformation*

$$(\tilde{*}) \quad \nabla(x,t) = d - \begin{pmatrix} 0 & 1 \\ Q(x) & 0 \end{pmatrix} \frac{dx}{x^{k+\mathbf{1}}},$$

with $Q(x)\left(\frac{dx}{x^{k+\mathbf{1}}}\right)^2$ in the normal form.

Corollary

Isomonodromic deformation is iso-Stokes.

Universal isomonodromic deformation (Heu (2010))

Given a meromorphic $\mathfrak{sl}_2(\mathbb{C})$-connection $\nabla_0(x)$ on $(E_0 \to X)$ with poles at $x_{i,0}$, $i = 1, \ldots, p$, of Poincaré ranks k_i and Katz ranks $k_i - \frac{m_i}{2}$.

Space of parameters of the universal isomonodromic deformation

$$T = \left\{(x_1, \ldots, x_p) \in X^p,\ x_i \neq x_j\right\} \times \Pi_{i=1}^{p} P_i \Big/ \mathrm{Aut}(X),$$

Local parameters at a singularity x_i:

$$P_i = \left\{ \begin{array}{c} \text{analytic coordinate-gauge} \\ \text{equivalence class of } \nabla_0 \text{ at } x_i \end{array} \right\} \Big/ \left[\begin{array}{c} \text{analytic gauge} \\ \text{equivalence} \end{array} \right]$$

$$= \left\{ \begin{array}{c} \text{analytic point} \\ \text{equivalence class of} \\ d - \begin{pmatrix} 0 & 1 \\ Q_{i,0}(x) & 0 \end{pmatrix} \frac{dx}{x^{k_i+1}} \end{array} \right\} \Big/ \left[\begin{array}{c} \text{analytic gauge} \\ \text{equivalence} \end{array} \right]$$

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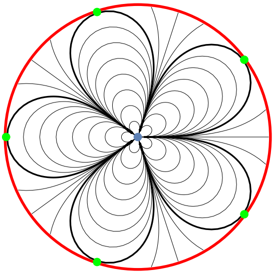
$$= \left\{ \begin{array}{c} \text{jets } \left(j^{n_i}\tilde{Q}(x)\right)\left(\frac{dx}{x^{k_i+1}}\right)^2 \\ \text{of quadratic differentials} \\ \text{equivalent to } Q_i(x)\left(\frac{dx}{x^{k_i+1}}\right)^2 \end{array} \right\}, \quad n_i = \min\left\{3k_i, \max\left\{\begin{matrix} k_i+m_i-1, \\ k_i+\lfloor\frac{m_i}{2}\rfloor \end{matrix}\right\}\right\}.$$

Stokes geometry

$Q(x)\left(\frac{dx}{x^{k+1}}\right)^2$ meromorphic quadratic differential, assume $m < 2k$ (irregular case).

Horizontal foliation: real time trajectories of

$$\frac{dx}{dt} = \pm\frac{x^{k+1}}{\sqrt{Q(x)}}, \quad t \in \mathbb{R}.$$



- *Complete real trajectories inside a disc* $\mathbb{D}$: organized in $2k - m$ *petals*.
- *Ends of petals* = tangent points with boundary $\partial\mathbb{D}$.
- *(anti)-Stokes regions*: trajectories that leave $\mathbb{D}$ (asymptotic to (anti)-Stokes directions at 0).

Stokes geometry

Theorem (Birkhoff, Hukuhara, Turittin, Sibuya)

System $x^{k+1}\frac{d}{dx}y = \begin{pmatrix} 0 & 1 \\ Q(x) & 0 \end{pmatrix} y$*:*

1. *Half-trajectory* $\leftrightsquigarrow$ *1-dimensional space of subdominant solutions (those with vanishing limit along the trajectory),*
2. *Complete trajectory* $\leftrightsquigarrow$ *direct decomposition of solutions space, sectorial basis: mixed solution basis on the petal.*

Stokes matrices: change of basis when crossing (anti)-Stokes regions

Stokes representation of fundamental groupoid:

$$\Pi_1(\overline{\mathbb{D}} \setminus \mathrm{Sing}, \mathrm{Ends}) \to \mathrm{SL}_2(\mathbb{C}), \qquad \mathrm{Sing} = \{0\},$$

a generalization of the monodromy representation.

Deformation of singularities

Up to analytic gauge transformation (analytic in parameter):

$$(\tilde{*}) \quad \epsilon^s P(x,\epsilon)\frac{dy}{dx} = \begin{pmatrix} 0 & 1 \\ Q(x,\epsilon) & 0 \end{pmatrix} y, \qquad P(x,0) = x^{k+1}, \qquad \epsilon \in (\mathbb{C}^N, 0).$$

In particular:

(o) local singular perturbations: $s \neq 0$,

(i) unfolding of singularities: $P(x,\epsilon) = x^{k+1} + \epsilon_k x^k + \ldots + \epsilon_0$,

(ii) unfolding of degeneracy: $P(x,\epsilon) = x^{k+1}$, $m > 0$,
$Q(x,\epsilon) = \epsilon'_0 + \ldots + \epsilon'_{m-1}x^{m-1} + q_m(\epsilon)x^m + \ldots$, $q_m(0) \neq 0$.

Example: *Painlevé equations*

Confluences of isomonodromic deformations

- isomonodromic parameter t,
- confluence parameter ϵ.

Type $\begin{cases} \text{(i) with } m = 0, \\ \text{(ii) with } m = 1, \end{cases}$ and $s = 0$.

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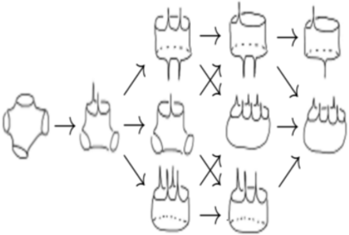

Confluent Stokes geometry

Meromorphic quadratic differential $Q(x,\epsilon)\left(\frac{dx}{P(x,\epsilon)}\right)^2$.

- *Saddles* = zeros and poles order 1 of $\frac{Q(x,\epsilon)}{P(x,\epsilon)^2}$.
- The *horizontal foliation* of

  $$e^{-2i\theta}Q(x,\epsilon)\left(\frac{dx}{P(x,\epsilon)}\right)^2, \qquad \text{angle of rotation} \quad -\tfrac{\pi}{2}+ < \theta < \tfrac{\pi}{2}-,$$

  is *rotationally stable* inside $\mathbb{D}$ if no trajectory leaves $\mathbb{D} \setminus$ Saddles in both directions.
- *Landing half-trajectories* $\leftrightsquigarrow$ 1-dimensional spaces of *subdominant solutions* (Levinson (1948)).

Theorem

If $2k - m > 0$, $|\epsilon|$ small enough, (ϵ, θ) s.t. the horizontal foliation is rotationally stable, and

- *either $m = 0$ (Hurtubise, Lambert, Rousseau (2014)),*
- *or $m = 1$ and $P(x,\epsilon) = x^{k+1}$ (K. (2020)),*
- *or ...*

Then, complete trajectories $\leftrightsquigarrow$ mixed solution bases,
when normalized they tend to the sectorial mixed solution bases at the limit $\epsilon \to 0$.

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Confluence

- *(Unfolded) Stokes matrices:* change of normalized mixed solution bases when crossing (anti)-Stokes regions between petals.
- *Stokes representation of the fundamental groupoid:* $\Pi_1(\overline{\mathbb{D}} \setminus \mathrm{Sing}(\epsilon), \mathrm{Ends}(\epsilon)) \to \mathrm{SL}_2(\mathbb{C})$.

Example (Hurtubise, Lambert, Rousseau (2014)):
$(x^3 + \epsilon_1 x + \epsilon_0)\frac{dy}{dx} = \begin{pmatrix} 0 & 1 \\ Q(x,\epsilon) & 0 \end{pmatrix} y, \qquad Q(0,0) \neq 0$ (non-resonance).

- 4 ends: 4 petals have limits when $\epsilon \to 0$

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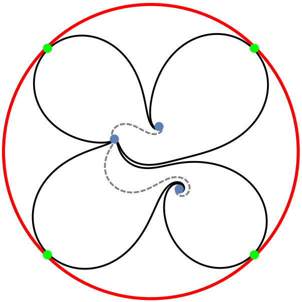
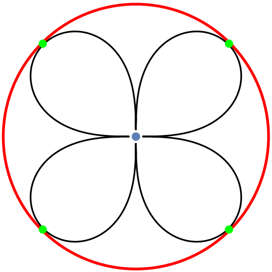


(a) $\epsilon \neq 0$

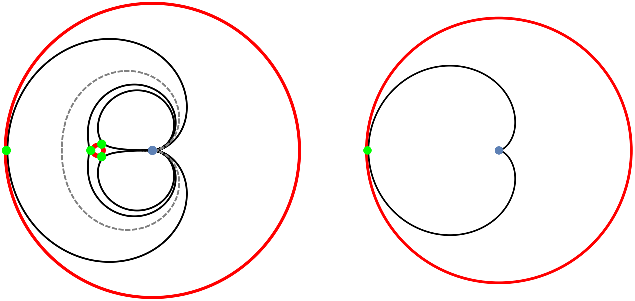


(b) $\epsilon = 0$

Confluence

Example (K. (2020)): $x^2 \frac{dy}{dx} = \begin{pmatrix} 0 & 1 \\ Q(x,\epsilon) & 0 \end{pmatrix} y, \qquad Q(x,\epsilon) = \epsilon + x + \dots.$

- Petals defined over some sector $\epsilon \in E$:
  - *saddle (turning point)* at $x \sim -\epsilon$ with 3 ends: 3 *inner petals* shrink and disappear as $\epsilon \to 0$,
  - 1 end on the boundary $\partial\mathbb{D}$: 1 *outer petal* persists to the limit.



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