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Complex dimensions and lengths of epsilon-neighborhoods of orbits

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Talk based on

P. Mardešić, G. Radunovic, M. Resman, *Fractal zeta functions of orbits of parabolic diffeomorphisms*, submitted (2020), https://arxiv.org/abs/2010.05955

Standard zeta function

- The *standard Hurwitz (Riemann) zeta function*

$$\zeta_a(s) := \sum_{j=0}^{\infty} \frac{1}{(j+a)^s}, \ a > 0, \ \text{Re}(s) > 1$$

  - converges absolutely for $\text{Re}(s) > 1$
  - meromorphically extendable to $\mathbb{C} \setminus \{1\}$
  - single pole at 1 with residue $\mathrm{Res}(\zeta_a(s), s = 1) = 1$
  - for $a = 1$: the Riemann zeta function

'Geometric generalizations' - **fractal zeta functions** in the sense of *Lapidus*

- $\mathcal{L} := \{\ell_j : j \in \mathbb{N}\}$
  a disjoint union of intervals on the real line with lengths ℓ_j

(1) The *geometric zeta function* of a *fractal string* (Lapidus, Frankenhuijsen, 2000)

$$\zeta_{\mathcal{L}}(s) := \sum_{j=1}^{\infty} \ell_j^s, \; s \in \mathbb{C} \text{ s.t. the sum converges absolutely}$$

$\star$ $\ell_j := \frac{1}{j}$ standard zeta function

Generalizations for arbitrary sets

(2) The *distance zeta function* of a bounded set $A \subseteq \mathbb{R}^N$

$$\zeta_A(s) := \int_{A_\delta} d(x, A)^{s-N}\, dx$$

- $\delta > 0$ inessential (up to a holomorphic function)

(3) The *tube zeta function* of a bounded set $A \subseteq \mathbb{R}^N$:

- the tube function of A:

$$\varepsilon \mapsto V_A(\varepsilon) := |A_\varepsilon| \text{ (the Lebesgue measure)}$$

- $V_A(\varepsilon) \sim M\varepsilon^{N-s_0},\ \varepsilon \to 0 \Rightarrow \dim_B(A) = s_0,\ \mathcal{M}^{s_0}(A) = M.$

$$\tilde{\zeta}_A(s) := \int_0^\delta t^{s-N-1} V_A(t)\, dt,$$

$$\mathrm{Re}(s) > \dim_B(A),\ \ \delta > 0 \text{ inessential}$$

(Lapidus, Frankenhuijsen 2000, 2006; Lapidus, Radunović, Žubrinić, 2017)

For fractal strings, all three equal up to a holomorphic function

$$\mathcal{L} \Rightarrow \; A := \{a_j : \; j \in \mathbb{N}_0\}, \; \ell_j := a_{j-1} - a_j$$

The functional equations on domains of definition (up to holomorphic functions):

- $\zeta_A(s) = \frac{2^{N-s}}{s}\zeta_{\mathcal{L}}(s)$,
- $\tilde{\zeta}_A(s) = \frac{1}{N-s}\zeta_A(s)$, $\mathrm{Re}(s) > \dim_B(A)$.

Definition

Let

- $A \subseteq \mathbb{R}^N$ *bounded,*
- $\zeta_A(s)$ *admits the meromorphic extension to whole* $\mathbb{C}$*.*

The set of all poles is called the set of complex dimensions of A*,* $\Omega(A)$*.*

- $\zeta_A(s)$ holomorphic for $\mathrm{Re}(s) > \dim_B(A)$,
- *simple pole at* $s = \dim_B(A)$.

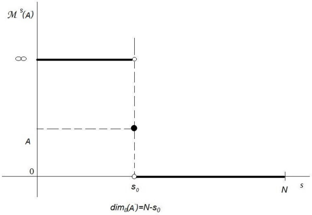
Complex dimensions (and their residues i.e. principal parts) *talk* about the geometry of the set! Similarly as the tube function!

Box dimension - the 'first' complex dimension of $A \subset \mathbb{R}^N$

The **box dimension** of a set (type of fractal dimension)

- $A \subseteq \mathbb{R}^N$ bounded
- $\varepsilon > 0$, $V_A(\varepsilon) := |A_\varepsilon|$ the Lebesque measure of the ε-neighborhood (the so-called *tube-function*)
- For $s \in [0, N]$, consider (or: $\liminf, \limsup$)

$$\lim_{\varepsilon \to 0} \frac{V_A(\varepsilon)}{\varepsilon^{N-s}} \in [0, \infty],$$



Slika: $s \mapsto \lim_{\varepsilon \to 0} \frac{V_A(\varepsilon)}{\varepsilon^{N-s}}$, $s \in [0, N]$.

- the moment of jump $s_0 \equiv$ the box dimension, $\dim_B(A) = s_0$.
- the value at $s_0 \equiv$ **Minkowski content**, $\mathcal{M}(A)$.
- if $V_A(\varepsilon) \sim_{\varepsilon \to 0} C\varepsilon^{N-s_0} \;\Rightarrow\; \dim_B(A) = s_0,\; \mathcal{M}(A) = C$.

Example 1

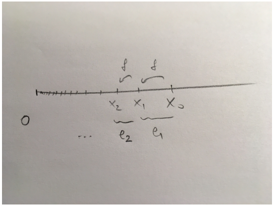
- $\dim_B(point) = 2 - 2 = 0;$
- $\dim_B(finite - length\ line) = 2 - 1 = 0;$
- $\dim_B(A) = 2 - 0 = 2,\; |A| > 0.$

Non-trivial examples: Orbits of local parabolic diffeomorphisms ($\equiv$ germs) on the real line $\mathbb{R}_+$

- (attracting) *parabolic* germ
  $f(x) = x - ax^{k+1} + \ldots \in \mathrm{Diff}(\mathbb{R}_+, 0)$, $a > 0$, $k \in \mathbb{N}$
  $$a_j \sim j^{-1/k},\ \ j \to \infty,$$
- (attracting) *hyperbolic* germ $f(x) = \lambda x + \ldots$, $0 < \lambda < 1$
  $$a_j \sim \lambda^j,\ \ j \to \infty.$$

Orbit of f with initial point $x_0 \in (\mathbb{R}_+, 0)$:

$$\mathcal{O}_f(x_0) := \{x_n := f^{\circ n}(x_0) :\ n \in \mathbb{N}_0\}$$



Box dimension and Minkowski content of orbits

Žubrinić, Županović 2005, MRŽ 2012

- a parabolic orbit of *multiplicity* k

$$V_{\mathcal{O}^f(x_0)}(\varepsilon) \sim (2/a)^{\frac{1}{k+1}} \frac{k+1}{k} \varepsilon^{\frac{1}{k+1}} + o(\varepsilon^{\frac{1}{k+1}}),\ \varepsilon \to 0,$$

$$\dim_B(\mathcal{O}^f(x_0)) = 1 - \frac{1}{k+1},\ \mathcal{M}(\mathcal{O}^f(x_0)) = (2/a)^{\frac{1}{k+1}} \frac{k+1}{k},$$

- a hyperbolic orbit

$$V_{\mathcal{O}^f(x_0)}(\varepsilon) \sim a(\lambda) \cdot \varepsilon(-\log \varepsilon) + o(\varepsilon(-\log \varepsilon)),\ \varepsilon \to 0,$$

$$\dim_B(\mathcal{O}^f(x_0)) = 1 - 1 = 0,\ \mathcal{M}(\mathcal{O}^f(x_0)) = +\infty.$$

Later: R [2013]
$*$ *formal class* of f using asymptotic expansion of function $\varepsilon \mapsto V_{\mathcal{O}^f(x_0)}(\varepsilon)$, as $\varepsilon \to 0$
$*$ further complex dimensions needed

Example 2 (The complex dimensions of the ternary Cantor set, LRŽ 2017)

$\star$ viewed as a fractal string, the order of intervals not important
$\mathcal{L}_{\mathcal{C}}$, A

$\zeta_{\mathcal{L}_{\mathcal{C}}}(s) = \sum_{j=1}^{\infty} \ell_j^s = \sum_{k=0}^{\infty} 2^k \left(\frac{1}{3^{k+1}}\right)^s = \frac{1}{3^s - 2}, \ |\frac{2}{3^s}| < 1$

- holomorphic for $\mathrm{Re}(s) > \log_2 3 = \dim_B \mathcal{C}$
- unique meromorphic extension to $\mathbb{C}$ by the above formula with poles:

$$\Omega(\mathcal{C}) = \{\omega_k := \log_3 2 + i\frac{2k\pi}{\log 3}, \ k \in \mathbb{Z}\}.$$

Example 3 (The tube function of the Cantor set (LRŽ 2017))

$$V_{\mathcal{C}}(\varepsilon) = \varepsilon^{1-\log_3 2}\big(G(-\log\varepsilon) + o(1)\big), \ \varepsilon \to 0,$$

G a nonconstant periodic function.

A conjecture (LRŽ):
Strong oscillations in the first term indication of self-similarity; non-real complex dimensions;
possible definition of *fractality* of a set as possessing non-real complex dimensions?

Complex dimensions deduced from asymptotics of the tube function of a set

(formally proven in LRŽ, 2017)

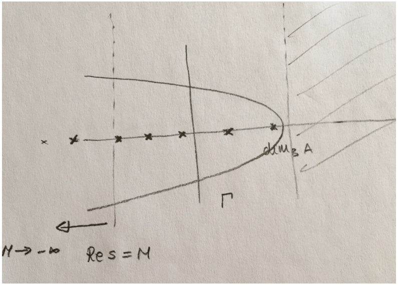
$\star$ $\tilde{\zeta}_A$ the tube zeta function of set $A \subseteq \mathbb{R}^N$, meromorphically extendable to $\mathbb{C}$.

$\star$ $t \mapsto V_A(t) = |A_t|$, $t \in (0, \delta)$, the tube function of A

- $\tilde{\zeta}_A(s) = \mathcal{M}(\chi_{(0,\delta)} V_A / \mathrm{id}^N)(s) = \int_0^\delta V_A(t) t^{s-1-N}\, dt$
- Conversely,

$$V_A(t) = \frac{t^N}{2\pi i} \mathcal{I}\mathcal{M}(\tilde{\zeta}_A)(t) = \frac{1}{2\pi i} \int_{\Gamma_c} \tilde{\zeta}_A(s) t^{N-s}\, ds, \;\; t \in (0, d).$$

Γ... a vertical line at around $s = c$, $\;c > \dim_B A$



$*$ Heuristically, the residue theorem 'gives' expansions of $t \mapsto V_A(t)$ from poles and residues of ζ_A:

$\star$ e.g. $\Omega_A = \{\omega_n,\ n \in \mathbb{N}\}$ only first-order poles

$$(**)\quad V_A(t) = \frac{1}{2\pi i} \int_{\Gamma_c} \frac{t^{N-s}}{N-s+1} \tilde{\zeta}_A(s)\, ds =$$

$$= \frac{1}{2\pi i} \sum_{\omega \in \Omega_A,\ \mathrm{Re}(\omega) > -M} \frac{t^{N-\omega}}{N-\omega+1} \mathrm{Res}(\tilde{\zeta}_A, \omega) + O(t^{N+M}),\ t \to 0,\ M \in \mathbb{N}.$$

(in case of higher-order poles logarithmic terms in the expansion)

Idea of proof of (**) (LRŽ)

- to get asymptotic remainder $O(t^{N+M})$, $M \in \mathbb{N}$, bounds needed on zeta function along vertical lines $\mathrm{Re}(s) = -M$, $M \to \infty$
- so-called *languidity bounds* of $\tilde{\zeta}_A(s)$ along vertical lines $s = \sigma + i\tau$, as $\tau \to \pm\infty$
- **pointwise asymptotics** as long as bounds *rational*

$$|\tilde{\zeta}_A(\sigma + i\tau)| \sim \tau^{-\gamma}, \ \gamma > 0, \tau \to \pm\infty$$

- *polynomial bounds* $(\gamma < 0) \Rightarrow$ only **distributional asymptotics** (there exists some primitive of tube function $t \mapsto V_A^{[k]}$) that expands pointwise up to this term, but differentiation of asymptotic expansions can be done just distributionally!
- $\frac{t^{N-s+k}}{(N-s+1)_k}\tilde{\zeta}_A(\sigma + i\tau)$, as $\tau \to \pm\infty$, becomes rational for k sufficiently big!

Relation to dynamical systems - 1-D orbits of germs seen as fractal strings

- (attracting) *parabolic* germ
  $f(z) = z - ax^{k+1} + \ldots \in \mathrm{Diff}(\mathbb{R}_+, 0)$, $a > 0$, $k \in \mathbb{N}$

$$a_j \sim j^{-1/k},\ \ell_j \sim j^{-\frac{k+1}{k}},\ j \to \infty,$$

- (attracting) *hyperbolic* germ $f(x) = \lambda x + \ldots$, $0 < \lambda < 1$

$$a_j \sim \lambda^j,\ \ell_j \sim \lambda^j,\ j \to \infty.$$

$$\zeta_{\mathcal{L}_f}(s)\text{''} \sim \text{''} \sum_{j \in \mathbb{N}} j^{-s\frac{k+1}{k}}$$

∗ holomorphic for $\mathrm{Re}(s) > \frac{k}{k+1} = \dim_B \mathcal{O}^f(x_0)$
∗ however, too coarse approximations for meromorphic extensions - info on poles and residues lost
∗ notation: $\zeta_{\mathcal{L}_f}$, ζ_f, $\tilde{\zeta}_f$

Precise computations tedious even in the simplest model case of germs, $k = 1$, $\rho = 0$ (MRR 2020)

$*$ *Model cases* with residual invariant $\rho = 0$ and multiplicity $k \in \mathbb{N}$
$*$ time-one maps of simple vector fields $x' = -x^{k+1}$:

$$f_k(x) := \mathrm{Exp}(x^{k+1}\frac{d}{dx}) = \frac{x}{(1+kx^k)^{1/k}} = x - x^{k+1} + o(x^{k+1}),\ k \in \mathbb{N}.$$

Heuristical proof in the simplest model case $k = 1, \ \rho = 0$

Putting $X := x_0^{-1}$,

$$\ell_j = \frac{1}{(j+X)(j+1+X)} = \frac{1}{(j+X)^2} \cdot \left(1 + \frac{1}{j+X}\right)^{-1},$$

$$\ell_j^s = \frac{1}{(j+X)^{2s}} \cdot \left(1 + \frac{1}{j+X}\right)^{-s} =$$

$$= \sum_{m=0} \binom{-s}{m} \frac{1}{(j+X)^{2s+m}}.$$

Heuristically (formal change of order of summation),

$$\zeta_{\mathcal{L}_{f_1}}(s) = \sum_{j=0} \ell_j^s \text{ " } \sim \text{ " } \sum_{m=0} \binom{-s}{m} \zeta_X(2s+m). \qquad (1)$$

Complex dimensions: $\omega_n := \frac{1-n}{2}, \ n \in \mathbb{N}_0$, with residues: $\operatorname{Res}(\zeta_{\mathcal{L}_{f_1}}, \omega_n) = \binom{\frac{n-1}{2}}{n}$. Zero residue for n odd.

What to do in the case $\rho \neq 0$ or even non-model case?

Arbitrary **parabolic** germ

$$f(x) = x - ax^{k+1} + o(x^{k+1}) \in \text{Diff}(\mathbb{R}_+, 0)$$

Theorem B (MRR 2020, Complex dimensions for arbitrary parabolic orbits)

$f \in \text{Diff}(R_+, 0)$, *of formal class* (k, ρ), $k \in \mathbb{N}$, $\rho \in \mathbb{R}$.

(1) *The distance zeta function* $\zeta_f(s)$ *can be meromorphically extended to* $\mathbb{C}$.

Theorem B

(2) *For* $s \in W_M := \{s > 1 - \frac{M}{k+1}\}$, $M \in \mathbb{N}$*:*

$$\zeta_f(s) = (1-s)\sum_{m=1}^{k} \frac{a_m}{s - \left(1 - \frac{m}{k+1}\right)} + (1-s)\left(\frac{b_{k+1}(x_0)}{s} + \frac{a_{k+1}}{s^2}\right) + $$

$$+ (1-s)\sum_{m=k+2}^{M-1} \sum_{p=0}^{\lfloor \frac{m}{k} \rfloor + 1} \frac{(-1)^p p! \cdot c_{m,p}(x_0)}{\left(s - \left(1 - \frac{m}{k+1}\right)\right)^{p+1}} + g(s),$$

$g(s)$ *holomorphic in* W_M*.*

$*$ the coefficients in principal parts of poles real, with dependence on x_0, as noted!

$*$ related to the coefficients of the asymptotic expansion of the tube function of the orbit!

$*$ **new** wrt model: **higher-order poles** correspond to *logarithmic terms* in the asymptotic expansion of the tube function due to $\rho \neq 0$

Hyperbolic orbits

* $\mathcal{O}_f(x_0) = \{x_0\lambda^n : n \in \mathbb{N}_0\}$,
* $\mathcal{L}_f := \{\ell_j := f^{\circ j}(x_0) - f^{\circ(j+1)}(x_0) = x_0(1-\lambda)\lambda^j : j \in \mathbb{N}_0\}$,
*
$$\zeta_f(s) := \frac{2^{1-s}}{s}\sum_{j=0}^{\infty}\ell_j^s = \frac{2^{1-s}x_0^s \cdot (1-\lambda)^s}{s}\frac{1}{1-\lambda^s},$$

* extends meromorphically from $\{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$ to $\mathbb{C}$: double pole $s_0 = 0$ and the simple poles

$$s_k := \frac{2k\pi}{\log\lambda}i,\ \ k \in \mathbb{Z}.$$

*
$$V_f(\varepsilon) = -\frac{2}{\log\lambda}\varepsilon(-\log\varepsilon) + H\Big(\log_\lambda\frac{2\varepsilon}{x_0(1-\lambda)}\Big)\cdot\varepsilon,$$

$H : [0,+\infty) \to \mathbb{R}$ a 1-periodic bounded function

Generalized asymptotic expansion of the tube function of orbits $\varepsilon \mapsto V_f(\varepsilon)$

$f(x) = x - ax^{k+1} + o(x^{k+1}) \in \mathrm{Diff}(\mathbb{R}, 0)$, $a > 0$, arbitrary parabolic germ

'Problem' noted in (R, 2014), (MRRZ, 2019):
$(*)$ the tube function $\varepsilon \mapsto V_f(\varepsilon)$ fails to have a full asymptotic expansion in power-logarithm scale,
$(*)$ oscillatory coefficient at order $O(\varepsilon^{\frac{2k+1}{k+1}}),\ \varepsilon \to 0.$

Generalized asymptotic expansion of the tube function of orbits $\varepsilon \mapsto V_f(\varepsilon)$

Proposition (MRR 2020)

A **generalized asymptotic expansion** *of the tube function with full description of* **oscillatory** *coefficients:*

$$V_f(\varepsilon) \sim 2^{\frac{1}{k+1}} a^{-\frac{1}{k+1}} \frac{k+1}{k} \cdot \varepsilon^{\frac{1}{k+1}} + \sum_{m=2}^{k} a_m \cdot \varepsilon^{\frac{m}{k+1}} + 2\rho \frac{k-1}{k} \cdot \varepsilon \log \varepsilon + b_{k+1}(x_0)\varepsilon +$$

$$+ \sum_{m=k+2}^{2k} \sum_{p=0}^{\lfloor \frac{m}{k} \rfloor + 1} c_{m,p} \varepsilon^{\frac{m}{k+1}} \log^p \varepsilon + \sum_{p=1}^{\lfloor \frac{2k+1}{k} \rfloor + 1} c_{2k+1,p} \varepsilon^{\frac{2k+1}{k+1}} \log^p \varepsilon +$$

$$+ \tilde{P}_{2k+1}(G(\tau_\varepsilon)) \cdot \varepsilon^{\frac{2k+1}{k+1}} + \sum_{m=2k+2}^{\infty} \sum_{p=0}^{\lfloor \frac{m}{k} \rfloor + 1} \tilde{Q}_{m,p}(G(\tau_\varepsilon)) \cdot \varepsilon^{\frac{m}{k+1}} \log^p \varepsilon, \ \varepsilon \to 0^+.$$

(∗) $\varepsilon \mapsto \tau_\varepsilon$ *the so-called* **continuous critical time** *(MRRZ 2019)*
(∗) $G : [0, +\infty) \to \mathbb{R}$ 1-**periodic**, $G(s) = 1 - s,\ s \in (0, 1),\ G(0) = 0$
(∗) $\tilde{P}_{2k+1}$ *resp.* $\tilde{Q}_{m,p}$, **polynomials** *whose coefficients in general depend on coefficients of* f *and initial condition* x_0.

Two ways to 'regularize' oscillatory coefficients:

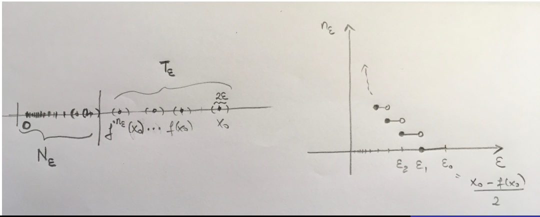
- the continuous-time tube function ('dynamical' regularization)
- the expansion of the tube function in the sense of Schwarz distributions

The continuous tube function-a *'dynamical smoothening'* of the expansion (MRRZ 2019)

$V_f(\varepsilon) = |O^f(x_0)_\varepsilon| = |T_\varepsilon| + |N_\varepsilon| = 2\varepsilon \cdot n_\varepsilon + (f^{n_\varepsilon}(x_0) + 2\varepsilon),\ \ \varepsilon > 0$
(the standard computation of the tube function: dividing into the *tail* and the *nucleus*)

$(*)$ $\varepsilon \mapsto n_\varepsilon$ the so-called **discrete critical time** 'separating' tail and nucleus - jump function at $\varepsilon_n = \frac{f^n(x_0) - f^{n+1}(x_0)}{2} \to 0,\ n \to \infty$.
$(*)$ $n_\varepsilon \in \mathbb{N}$ determined by two inequalities:

$$|f^{n_\varepsilon - 1}(x_0) - f^{n_\varepsilon}(x_0)| \geq 2\varepsilon,$$
$$|f^{n_\varepsilon}(x_0) - f^{n_\varepsilon + 1}(x_0)| < 2\varepsilon.$$



$(*)$ the **continuous critical time** (MRRZ 2019) $\varepsilon \mapsto \tau_\varepsilon$
- an analytic, dynamical 'approximation' of n_ε
- relies on *embedding of f as the time-one map in a flow* $\{f^t:\ t\in\mathbb{R}\}$:

$$f^{\tau_\varepsilon}(x_0) - f^{\tau_\varepsilon+1}(x_0) = 2\varepsilon.$$

Note: $n_\varepsilon = \lfloor \tau_\varepsilon \rfloor + 1$, $\varepsilon > 0$
More precisely, $n_\varepsilon = \tau_\varepsilon + G(\tau_\varepsilon)$.

The **continuous tube function** $\varepsilon \mapsto V_f^c(\varepsilon)$

$$V_f^c(\varepsilon) = 2\varepsilon\tau_\varepsilon + (f^{\tau_\varepsilon}(x_0) + 2\varepsilon),\ \varepsilon > 0.$$

$(*)$ analytic in $\varepsilon \in (0,\delta)$
$(*)$ expansion coincides with the expansion of $\varepsilon \mapsto V_f(\varepsilon)$ up to the first oscillatory term
$(*)$ full asymptotic expansion in a power-log scale, no oscillatory coefficients!

Asymptotic expansion of the continuous tube function

f parabolic

Proposition (MRR 2020)

$$V_f^c(\varepsilon) \sim 2^{\frac{1}{k+1}} a^{-\frac{1}{k+1}} \frac{k+1}{k} \cdot \varepsilon^{\frac{1}{k+1}} + \sum_{m=2}^{k} a_m \cdot \varepsilon^{\frac{m}{k+1}} + 2\rho \frac{k-1}{k} \cdot \varepsilon \log \varepsilon +$$
$$+ b_{k+1}(x_0)\varepsilon + \sum_{m=k+2}^{\infty} \sum_{p=0}^{\lfloor \frac{m}{k} \rfloor + 1} c_{m,p} \varepsilon^{\frac{m}{k+1}} \log^p \varepsilon, \ \varepsilon \to 0^+.$$

$(*)$ $c_{2k+1,0}$ *resp.* $c_{m,p}$, $m \geq 2k+2$, $p = 0, \ldots, \lfloor \frac{m}{k} \rfloor + 1$, *are free coefficients of polynomials* $\tilde{P}_{2k+1}$ *resp.* $\tilde{Q}_{m,p}$

$(*)$ *only the coefficient* $b_{k+1}(x_0)$ *depends on the initial condition* x_0.

Distributional expansion: *distributional smoothening* of the tube function

f parabolic

Proposition (MRR 2020)

$$V_f(\varepsilon) \sim_{\mathcal{D}} 2^{\frac{1}{k+1}} a^{-\frac{1}{k+1}} \frac{k+1}{k} \cdot \varepsilon^{\frac{1}{k+1}} + \sum_{m=2}^{k} a_m \cdot \varepsilon^{\frac{m}{k+1}} + 2\rho \frac{k-1}{k} \cdot \varepsilon \log \varepsilon + b_{k+1}(x_0)\varepsilon +$$

$$+ \sum_{m=k+2}^{2k} \sum_{p=0}^{\lfloor \frac{m}{k} \rfloor + 1} c_{m,p} \varepsilon^{\frac{m}{k+1}} \log^p \varepsilon + \sum_{p=1}^{\lfloor \frac{2k+1}{k} \rfloor + 1} c_{2k+1,p} \varepsilon^{\frac{2k+1}{k+1}} \log^p \varepsilon +$$

$$+ d_{2k+1,0}(x_0) \cdot \varepsilon^{\frac{2k+1}{k+1}} + \sum_{m=2k+2}^{\infty} \sum_{p=0}^{\lfloor \frac{m}{k} \rfloor + 1} d_{m,p}(x_0) \cdot \varepsilon^{\frac{m}{k+1}} \log^p \varepsilon, \ \varepsilon \to 0^+.$$

Here,
$d_{2k+1,0}(x_0) := \int_0^1 \tilde{P}_{2k+1}(s)\, ds,$
$d_{m,p}(x_0) := \int_0^1 \tilde{Q}_{m,p}(s)\, ds, \ m \geq 2k+2, \ p = 0, \ldots, \lfloor \frac{m}{k} \rfloor + 1,$
the mean values of 1*-periodic functions* $\tilde{P}_{2k+1} \circ G$ *and* $\tilde{Q}_{m,p} \circ G$.

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Thank you for your attention!